Higher-Order Functions
Class outline:

- Iteration example
- Designing functions
- Generalization
- Higher-order functions
- Lambda expressions
- Conditional expressions
Iteration example
Virahaṅka-Fibonacci numbers

Discovered by Virahanka in India, 600–800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

0 1 1 2 3 5 8 13 21 34  ...
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0 + 1 = 1  2  3  5  8  13  21  34  ...

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\]
Virahaṇka-Fibonacci numbers

Discovered by Virahanka in India, 600-800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

0  1  1  2  3  5  8  13 + 21 = 34  ...
Virahanka's question

How many poetic meters exist for a total duration?

S = short syllable, L = long syllable

<table>
<thead>
<tr>
<th>Duration</th>
<th>Meters</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>SS, L</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>SSS, SL, LS</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>SSSS, SSL, SLS, LSS, LL</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>SSSSS, SSSL, SSLS, SLSS, SLL, LLS, LSL, LSSS</td>
<td>8</td>
</tr>
</tbody>
</table>

The So-called Fibonacci Numbers in Ancient and Medieval India
Fibonacci's question

How many pairs of rabbits can be bred after N months?
<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Σ</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Attribution: Fschwarzentruber, Wikipedia
# Virahanka-Fibonacci number generation

| VF | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | ...
|----|--|--|--|--|--|--|--|--|--|--|--|---|
| N  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ...

```python
def vf_number(n):
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> vf_number(2)
    1
    >>> vf_number(6)
    8
    """
    prev = 0  # First Fibonacci number
    curr = 1  # Second Fibonacci number
    k = 1
    while k < n:
        (prev, curr) = (curr, prev + curr)
        k += 1
    return curr
```
Golden spiral

The Golden spiral can be approximated by Virahanka-Fibonacci numbers.
Go bears!

The Golden spiral is found everywhere in nature...
Designing Functions
# Describing Functions

```python
def square(x):
    """Returns the square of X.""
    return x * x
```

<table>
<thead>
<tr>
<th>Aspect</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function's <strong>domain</strong> is the set of all inputs it might possibly take as arguments.</td>
<td>x is a number</td>
</tr>
<tr>
<td>A function's <strong>range</strong> is the set of output values it might possibly return.</td>
<td>square returns a non-negative real number</td>
</tr>
<tr>
<td>A pure function's <strong>behavior</strong> is the relationship it creates between input and output.</td>
<td>square returns the square of x</td>
</tr>
</tbody>
</table>
Designing a function

Give each function exactly one job, but make it apply to many related situations.

```python
round(1.23)    # 1
round(1.23, 0) # 1
round(1.23, 1) # 1.2
round(1.23, 5) # 1.23
```

**Don't Repeat Yourself (DRY):** Implement a process just once, execute it many times.
Generalization
Generalizing patterns with arguments

Geometric shapes have similar area formulas.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 \times r^2$</td>
</tr>
<tr>
<td></td>
<td>$\pi \times r^2$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3\sqrt{3}}{2} \times r^2$</td>
</tr>
</tbody>
</table>
A non-generalized approach

```python
from math import pi, sqrt

def area_square(r):
    return r * r

def area_circle(r):
    return r * r * pi

def area_hexagon(r):
    return r * r * (3 * sqrt(3) / 2)
```

How can we generalize the common structure?
from math import pi, sqrt

def area(r, shape_constant):
    """Return the area of a shape from length measurement R."""
    if r < 0:
        return 0
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3) / 2)
Higher-order functions
What are higher-order functions?

A function that either:

- Takes another function as an argument
- Returns a function as its result

All other functions are considered first-order functions.
Generalizing over computational processes

\[ \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15 \]

\[ \sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 \]

\[ \sum_{k=1}^{5} \frac{8}{(4k - 3) \cdot (4k - 1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \]

The common structure among functions may be a computational process, not just a number.
def cube(k):
    return k ** 3

def summation(n, term):
    """Sum the first N terms of a sequence."
    >>> summation(5, cube)
    225
    """
    total = 0
    k = 1
    while k <= n:
        total = total + term(k)
        k = k + 1
    return total
Functions as return values
Locally defined functions

Functions defined within other function bodies are bound to names in a local frame.

def make_adder(n):
    """Return a function that takes one argument k and returns k + n.
    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
Call expressions as operator expressions

make_adder(1)( 2 )

Operator 2  Operand
Call expressions as operator expressions

\[
\text{make_adder}(1)(\underline{\text{Operator}}\ 2\ \underline{\text{Operand}})
\]

\[
\text{make_adder}(1)\ \underline{\text{Operator}}\ 2\ \underline{\text{Operand}}
\]
Call expressions as operator expressions

\[
\text{make\_adder}(1)\left(\quad 2 \quad\right)
\]

\[
\text{func make\_adder...}
\]
Call expressions as operator expressions

\[
\text{make\_adder}(1)( \underline{\quad 2 \quad} )
\]

\[
\text{make\_adder}(1)
\]

\[
\text{func make\_adder... } 1
\]
Call expressions as operator expressions

\[
\text{make_adder}(1)(\quad 2 \quad)
\]

\[
\text{make_adder}(1)
\]

\[
\text{func make_adder...}
\]

\[
\text{def adder}(k):
    \text{return } k + n \\
\text{return adder}
\]
Call expressions as operator expressions

\[
\text{make_adder}(1)(2)
\]

\text{func adder(k)}

\text{make_adder}(1)

\text{func make_adder...}

\text{1}

\text{make_adder(n)}

\text{def adder(k):}
  \text{return k + n}
\text{return adder

\text{fur}
Call expressions as operator expressions

\[
\text{make\_adder}(1)(\ 2\ )
\]

\[
\text{func}\ \text{adder}(k)
\]

\[
\text{func}\ \text{make\_adder}...
\]

\[
\text{def}\ \text{adder}(k):
\]
\[
\text{return}\ k + n
\]
\[
\text{return}\ \text{adder}
\]

\[
\text{fur}
\]
Call expressions as operator expressions

```
def adder(k):
    return k + n
return adder
```

```
func make_adder...
```

```
func adder(k)
make_adder(1)
```

```
make_adder(1)(
    2
)
```

```
3
```

```
1
```

```
2
```

```
23
```
Lambda expressions
Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

```python
lambda <parameters>: <expression>
```

A function that takes in **parameters** and returns the result of **expression**.
Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

```
lambda <parameters>: <expression>
```

A function that takes in **parameters** and returns the result of **expression**.

A lambda version of the **square** function:

```
square = lambda x: x * x
```

A function that takes in parameter **x** and returns the result of **x * x**.
Lambda syntax tips

A lambda expression does not contain return statements or any statements at all.

Incorrect:

```python
square = lambda x: return x * x
```

Correct:

```python
square = lambda x: x * x
```
Def statements vs. Lambda expressions

def square(x):
    return x * x

vs

square = lambda x: x * x

Both create a function with the same domain, range, and behavior.

Both bind that function to the name square.

Only the def statement gives the function an intrinsic name, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).
Lambda as argument

It's convenient to use a lambda expression when you are passing in a simple function as an argument to another function.

Instead of...

def cube(k):
    return k ** 3

summation(5, cube)

We can use a lambda:

summation(5, lambda k: k ** 3)
Conditional expressions
Conditional expressions

A conditional expression has the form:

<consequent> if <predicate> else <alternative>

Evaluation rule:

• Evaluate the <predicate> expression.
• If it's a true value, the value of the whole expression is the value of the <consequent>.
• Otherwise, the value of the whole expression is the value of the <alternative>.
Lambdas with conditionals

This is invalid syntax:

```python
lambda x: if x > 0: x else: 0
```

Conditional expressions to the rescue!

```python
lambda x: x if x > 0 else 0
```