Higher-Order Functions
Class outline:

• Iteration example
• Designing functions
• Generalization
• Higher-order functions
• Lambda expressions
• Conditional expressions
Iteration example
Virahaṅka-Fibonacci numbers

Discovered by Virahanka in India, 600-800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

0 1 1 2 3 5 8 13 21 34 ...
Virahañka-Fibonacci numbers

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\[ 0 + 1 = 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \ldots \]
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0  1  +  1  =  2  3  5  8  13  21  34  ...
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\[
\begin{align*}
0 & \quad 1 & \quad 1 & \quad 2 & \quad 3 & \quad 5 & \quad 8 & \quad 13 & + 13 &= 21 & \quad 34 & \quad \ldots
\end{align*}
\]
Virahaṅka-Fibonacci numbers

Discovered by Virahanka in India, 600-800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

0 1 1 2 3 5 8 13 + 21 = 34 ...
Virahanka's question

How many poetic meters exist for a total duration?

S = short syllable, L = long syllable

<table>
<thead>
<tr>
<th>Duration</th>
<th>Meters</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>SS, L</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>SSS, SL, LS</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>SSSS, SSL, SLS, LSS, LL</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>SSSSS, SSSL, SSLS, SLSS, SLL, LLS, LSL, LSSS</td>
<td>8</td>
</tr>
</tbody>
</table>

The So-called Fibonacci Numbers in Ancient and Medieval India
Fibonacci's question

How many pairs of rabbits can be bred after N months?
<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Σ</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Attribution: Fschwarzentruber, Wikipedia
Virahanka-Fibonacci number generation

<table>
<thead>
<tr>
<th>VF</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>13</th>
<th>21</th>
<th>34</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

```python
def vf_number(n):
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> vf_number(2)
    1
    >>> vf_number(6)
    8
    """
    prev = 0  # First Fibonacci number
    curr = 1  # Second Fibonacci number
    k = 1
    while k < n:
        (prev, curr) = (curr, prev + curr)
        k += 1
    return curr
```

```bash
>>> vf_number(2)
1
>>> vf_number(6)
8
```

Golden spiral

The Golden spiral can be approximated by Virahanka-Fibonacci numbers.
Go bears!

The Golden spiral is found everywhere in nature...
Designing Functions
Describing Functions

```python
def square(x):
    """Returns the square of X.""
    return x * x
```

<table>
<thead>
<tr>
<th>Aspect</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function's <strong>domain</strong> is the set of all inputs it might possibly take as arguments.</td>
<td>$x$ is a number</td>
</tr>
<tr>
<td>A function's <strong>range</strong> is the set of output values it might possibly return.</td>
<td><code>square</code> returns a non-negative real number</td>
</tr>
<tr>
<td>A pure function's <strong>behavior</strong> is the relationship it creates between input and output.</td>
<td><code>square</code> returns the square of $x$</td>
</tr>
</tbody>
</table>
Designing a function

Give each function exactly one job, but make it apply to many related situations.

round(1.23)    # 1
round(1.23, 0) # 1
round(1.23, 1) # 1.2
round(1.23, 5) # 1.23

**Don't Repeat Yourself (DRY):** Implement a process just once, execute it many times.
Generalization
Generalizing patterns with arguments

Geometric shapes have similar area formulas.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 \times r^2$</td>
</tr>
<tr>
<td></td>
<td>$\pi \times r^2$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3\sqrt{3}}{2} \times r^2$</td>
</tr>
</tbody>
</table>
from math import pi, sqrt

def area_square(r):
    return r * r

def area_circle(r):
    return r * r * pi

def area_hexagon(r):
    return r * r * (3 * sqrt(3) / 2)

How can we generalize the common structure?
from math import pi, sqrt

def area(r, shape_constant):
    """Return the area of a shape from length measurement $R$."""
    if r < 0:
        return 0
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3) / 2)
Higher-order functions
What are higher-order functions?

A function that either:

• Takes another function as an argument
• Returns a function as its result

All other functions are considered first-order functions.
Generalizing over computational processes

\[ \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15 \]

\[ \sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 \]

\[ \sum_{k=1}^{5} \frac{8}{(4k - 3)(4k - 1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \]

The common structure among functions may be a computational process, not just a number.
def cube(k):
    return k ** 3

def summation(n, term):
    """Sum the first N terms of a sequence.
    >>> summation(5, cube)
    225
    """
    total = 0
    k = 1
    while k <= n:
        total = total + term(k)
        k = k + 1
Functions as return values
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.
    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
Call expressions as operator expressions

\[
\text{make_adder(1)}( \quad \quad 2 \quad \quad )
\]

Operator

Operand
Call expressions as operator expressions

\[
\text{make_adder(1)}( \quad 2 \quad )
\]

\[
\text{Operator} \quad \text{Operand}
\]

\[
\text{make_adder(1)}
\]
Call expressions as operator expressions

\[ \text{make\_adder}(1)(\quad 2\quad ) \]

\[ \text{make\_adder}(1) \]

\[ \text{func make\_adder...} \]
Call expressions as operator expressions

make_adder(1)( 1 2 )

Operator

make_adder(1)

func make_adder... 1

Operand
Call expressions as operator expressions

```
def adder(k):
    return k + n
return adder
```

```
make_adder(1)( 2 )
```

```
func make_adder...
```

```
1
```
Call expressions as operator expressions

```
def adder(k):
    return k + n
return adder
```

```
make_adder(1) ( 2 )
make_adder(1)

func make_adder...

func adder(k)
```

```
make_adder(n)
```
Call expressions as operator expressions

\[
\text{make\_adder}(1)(\text{ 2 })
\]

```python
def adder(k):
    return k + n
return adder
```

func make\_adder... 1
Call expressions as operator expressions

make_adder(1)( 2 )

func adder(k)

func make_adder...

make_adder(n)
def adder(k):
    return k + n
return adder

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Lambda expressions
Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

```python
lambda <parameters>: <expression>
```

A function that takes in **parameters** and returns the result of **expression**.
Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

`lambda <parameters>: <expression>`

A function that takes in **parameters** and returns the result of **expression**.

A lambda version of the **square** function:

```
square = lambda x: x * x
```

A function that takes in parameter `x` and returns the result of `x * x`.
Lambda syntax tips

A lambda expression does **not** contain return statements or any statements at all.

Incorrect:

```
square = lambda x: return x * x
```

Correct:

```
square = lambda x: x * x
```
Def statements vs. Lambda expressions

<table>
<thead>
<tr>
<th>def square(x):</th>
<th>vs</th>
<th>square = lambda x: x * x</th>
</tr>
</thead>
<tbody>
<tr>
<td>return x * x</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Both create a function with the same domain, range, and behavior.

Both bind that function to the name square.

Only the `def` statement gives the function an **intrinsic name**, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).
Lambda as argument

It's convenient to use a lambda expression when you are passing in a simple function as an argument to another function.

Instead of...

```python
def cube(k):
    return k ** 3

summation(5, cube)
```

We can use a lambda:

```python
summation(5, lambda k: k ** 3)
```
Conditional expressions
Conditional expressions

A conditional expression has the form:

<consequent> if <predicate> else <alternative>

Evaluation rule:

• Evaluate the <predicate> expression.
• If it's a true value, the value of the whole expression is the value of the <consequent>.
• Otherwise, the value of the whole expression is the value of the <alternative>.
Lambdas with conditionals

This is invalid syntax:

\[
\text{lambda } x: \text{ if } x > 0: x \text{ else: } 0
\]

Conditional expressions to the rescue!

\[
\text{lambda } x: x \text{ if } x > 0 \text{ else } 0
\]