Recursion
Recursive Functions
Recursive Functions

**Definition:** A function is called recursive if the body of that function calls itself, either directly or indirectly.

**Implication:** Executing the body of a recursive function may require applying that function.

Drawing Hands, by M. C. Escher (lithograph, 1948)
Digit Sums

If a number $a$ is divisible by 9, then $\text{sum_digits}(a)$ is also divisible by 9.

Useful for typo detection!

$2+0+1+9 = 12$

Credit cards actually use the Luhn algorithm, which we'll implement after $\text{sum_digits}$. 

A checksum digit is a function of all the other digits; it can be computed to detect typos.
The Problem Within the Problem

The sum of the digits of 6 is 6.

Likewise for any one-digit (non-negative) number (i.e., < 10).

The sum of the digits of 2019 is

That is, we can break the problem of summing the digits of 2019 into a smaller instance of the same problem, plus some extra stuff.

We call this recursion
def split(n):
    """Split positive n into all but its last digit and its last digit."""
    return n // 10, n % 10

def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
The Anatomy of a Recursive Function

• The def statement header is similar to other functions
• Conditional statements check for base cases
• Base cases are evaluated without recursive calls
• Recursive cases are evaluated with recursive calls

```python
def sum_digits(n):
    """Return the sum of the digits of positive integer n.""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

(Demo)
Recursion in Environment Diagrams
Recursion in Environment Diagrams

- The same function `fact` is called multiple times.
- Different frames keep track of the different arguments in each call.
- What `n` evaluates to depends upon the current environment.
- Each call to `fact` solves a simpler problem than the last: smaller `n`.
Iteration vs Recursion

Iteration is a special case of recursion

\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]

Using while:

```python
def fact_iter(n):
    total, k = 1, 1
    while k <= n:
        total, k = total*k, k+1
    return total
```

Using recursion:

```python
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Math:

\[ n! = \prod_{k=1}^{n} k \]

Names: \( n, \text{total}, k, \text{fact}_\text{iter} \)

\[ n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n - 1)! & \text{otherwise} \end{cases} \]

Names: \( n, \text{fact} \)
Verifying Recursive Functions
The Recursive Leap of Faith

```python
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Is fact implemented correctly?

1. Verify the base case

2. Treat `fact` as a functional abstraction!

3. Assume that `fact(n-1)` is correct

4. Verify that `fact(n)` is correct
Mutual Recursion
The Luhn Algorithm

Used to verify credit card numbers


• First: From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g., 7 * 2 = 14), then sum the digits of the products (e.g., 10: 1 + 0 = 1, 14: 1 + 4 = 5)

• Second: Take the sum of all the digits

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<td>2</td>
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<td>1+6=7</td>
<td>7</td>
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= 30

The Luhn sum of a valid credit card number is a multiple of 10 (Demo)
Recursion and Iteration
Converting Recursion to Iteration

Can be tricky: Iteration is a special case of recursion.

Idea: Figure out what state must be maintained by the iterative function.

```python
def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

(Demo)
Converting Iteration to Recursion

More formulaic: Iteration is a special case of recursion.

Idea: The state of an iteration can be passed as arguments.

```python
def sum_digits_iter(n):
    digit_sum = 0
    while n > 0:
        n, last = split(n)
        digit_sum = digit_sum + last
    return digit_sum

def sum_digits_rec(n, digit_sum):
    if n == 0:
        return digit_sum
    else:
        n, last = split(n)
        return sum_digits_rec(n, digit_sum + last)
```

Updates via assignment become... arguments to a recursive call
Order of Recursive Calls
The Cascade Function

- Each cascade frame is from a different call to `cascade`.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
    print(n)
```

Program output:

```
123
12
1
12
```
Two Definitions of Cascade

(Demo)

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

```python
def cascade(n):
    print(n)
    if n >= 10:
        cascade(n//10)
        print(n)
```

- If two implementations are equally clear, then shorter is usually better
- In this case, the longer implementation is more clear (at least to me)
- When learning to write recursive functions, put the base cases first
- Both are recursive functions, even though only the first has typical structure
Example: Inverse Cascade
Inverse Cascade

Write a function that prints an inverse cascade:

```python
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(lambda: grow(n), lambda: print(n), n // 10)
shrink = lambda n: f_then_g(lambda: shrink(n), lambda: print(n), n // 10)
```