Recursion

Announcements

Abstraction

Functional Abstractions

```
def square(x):
    return mul(x, x)
```

```
def sum_squares(x, y):
    return square(x) + square(y)
```

What does sum_squares need to know about square?

- Square takes one argument. **Yes**
- Square has the intrinsic name square. **No**
- Square computes the square of a number. **Yes**
- Square computes the square by calling mul. **No**

```
def square(x):
    return pow(x, 2)
```

```
def square(x):
    return mul(x, x-1) + x
```

If the name “square” were bound to a built-in function, sum_squares would still work identically.

Choosing Names

Names typically don't matter for correctness

but

they matter a lot for composition

From:	**To:**
true_false	rolled_a_one
d	dice
helper	take_turn
my_int	num_rolls
l, I, O	k, i, m

Names should convey the meaning or purpose of the values to which they are bound.

The type of value bound to the name is best documented in a function's docstring.

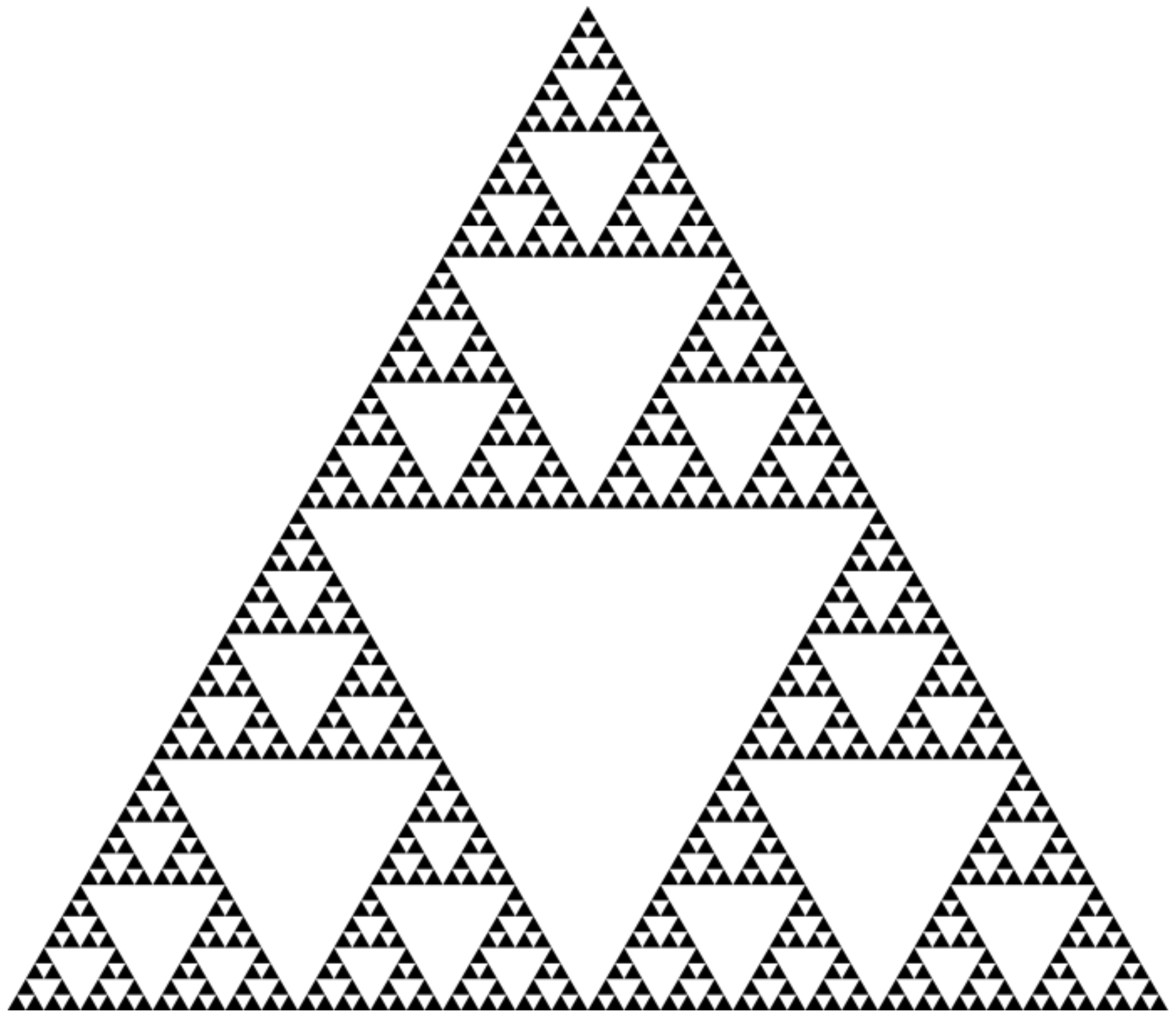
Function names typically convey their effect (**print**), their behavior (**triple**), or the value returned (**abs**).

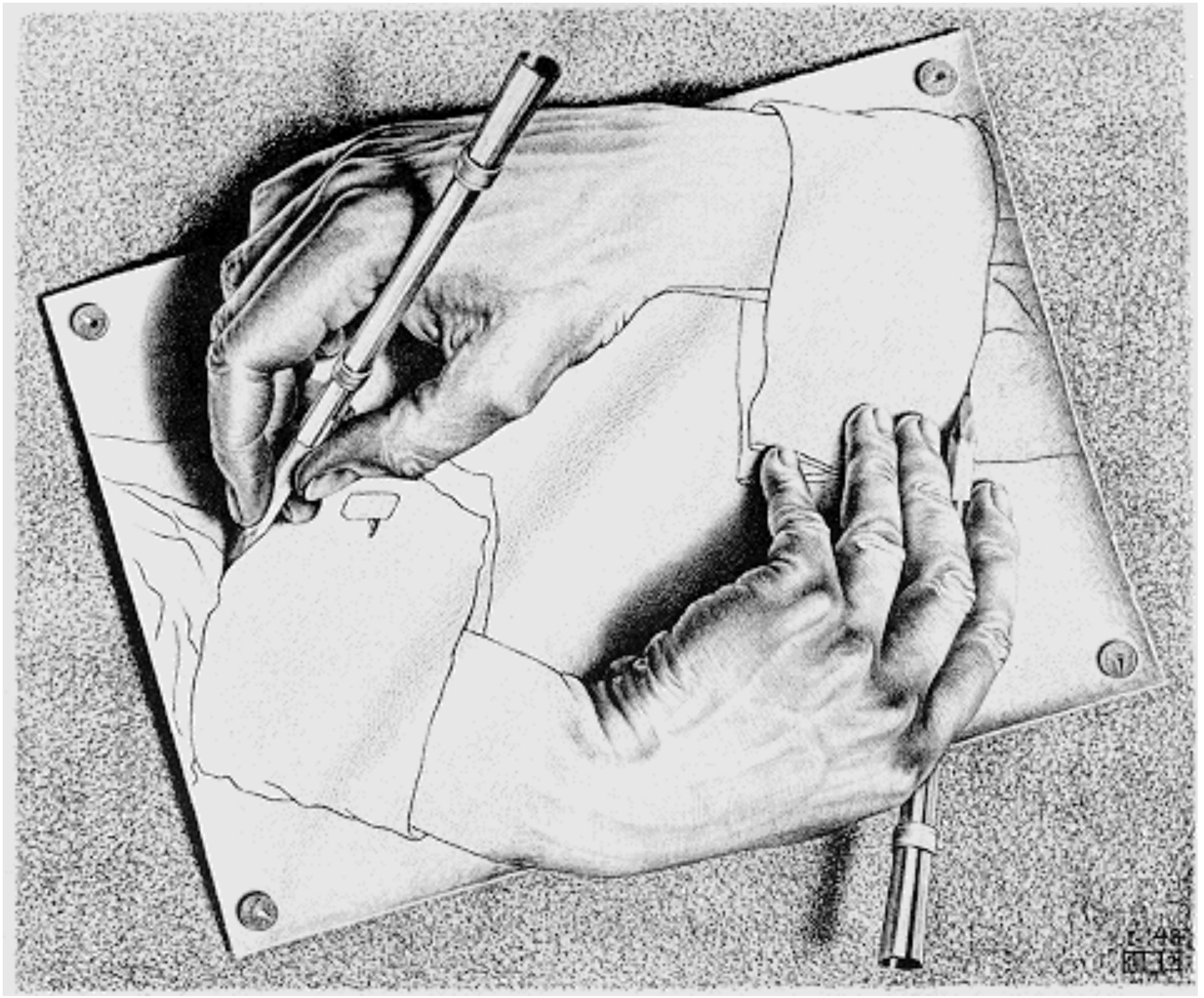
Recursive Functions

Recursive Functions

Definition: A function is called recursive if the body of that function calls itself, either directly or indirectly

Implication: Executing the body of a recursive function may require applying that function





Drawing Hands, by M. C. Escher (lithograph, 1948)

Digit Sums

2+0+1+9 = 12

- If a number a is divisible by 9, then `sum_digits(a)` is also divisible by 9
- Useful for typo detection!

A checksum digit is a function of all the other digits; It can be computed to detect typos

- Credit cards actually use the Luhn algorithm, which we'll implement after `sum_digits`

The Problem Within the Problem

The sum of the digits of 6 is 6.

Likewise for any one-digit (non-negative) number (i.e., < 10).

The sum of the digits of 2019 is

201 9

Sum of these digits + This digit

That is, we can break the problem of summing the digits of 2019 into a smaller instance of the same problem, plus some extra stuff.

We call this recursion

Sum Digits Without a While Statement

```
def split(n):
    """Split positive n into all but its last digit and its last digit."""
    return n // 10, n % 10

def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

The Anatomy of a Recursive Function

- The def statement header is similar to other functions
- Conditional statements check for base cases
- Base cases are evaluated without recursive calls
- Recursive cases are evaluated with recursive calls

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

(Demo)

Recursion in Environment Diagrams

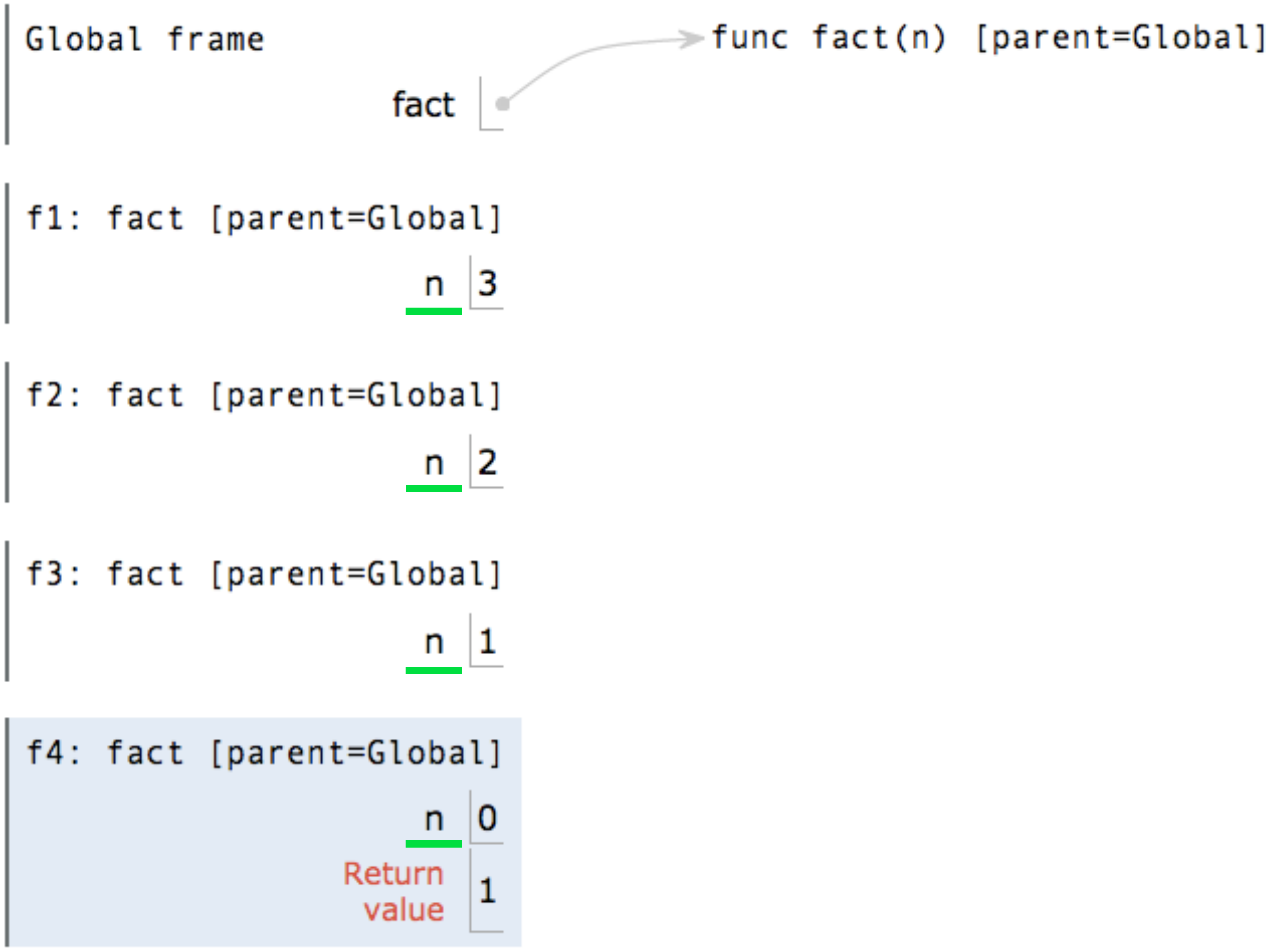
Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

- The same function **fact** is called multiple times
- Different frames keep track of the different arguments in each call
- What **n** evaluates to depends upon the current environment
- Each call to **fact** solves a simpler problem than the last: smaller **n**

(Demo)



Iteration vs Recursion

Iteration is a special case of recursion

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Using while:

```
def fact_iter(n):
    total, k = 1, 1
    while k <= n:
        total, k = total*k, k+1
    return total
```

Using recursion:

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Math:

$$n! = \prod_{k=1}^{n} k$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

Names:

n, total, k, fact_iter

n, fact

Verifying Recursive Functions

The Recursive Leap of Faith

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Is fact implemented correctly?

1. Verify the base case

2. Treat **fact** as a functional abstraction!

3. Assume that **fact(n-1)** is correct

4. Verify that **fact(n)** is correct

Photo by Kevin Lee, Preikestolen, Norway

Mutual Recursion

The Luhn Algorithm

Used to verify credit card numbers

From Wikipedia: http://en.wikipedia.org/wiki/Luhn_algorithm

- **First:** From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g., 7 * 2 = 14), then sum the digits of the products (e.g., 10: 1 + 0 = 1, 14: 1 + 4 = 5)
- **Second:** Take the sum of all the digits

1	3	8	7	4	3	
2	3	1+6=7	7	8	3	= 30

The Luhn sum of a valid credit card number is a multiple of 10 (Demo)

Recursion and Iteration

Converting Recursion to Iteration

Idea: Figure out what state must be maintained by the iterative function.

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

A partial sum

What's left to sum

(Demo)

Converting Iteration to Recursion

Idea: The state of an iteration are passed as arguments.

```
def sum_digits_iter(n):
    digit_sum = 0
    while n > 0:
        n, last = split(n)
        digit_sum = digit_sum + last
    return digit_sum
```

Updates via assignment become...

```
def sum_digits_rec(n, digit_sum):
    if n > 0:
        n, last = split(n)
        return sum_digits_rec(n,
    else:
        return digit_sum
```

...arguments to a recursive call

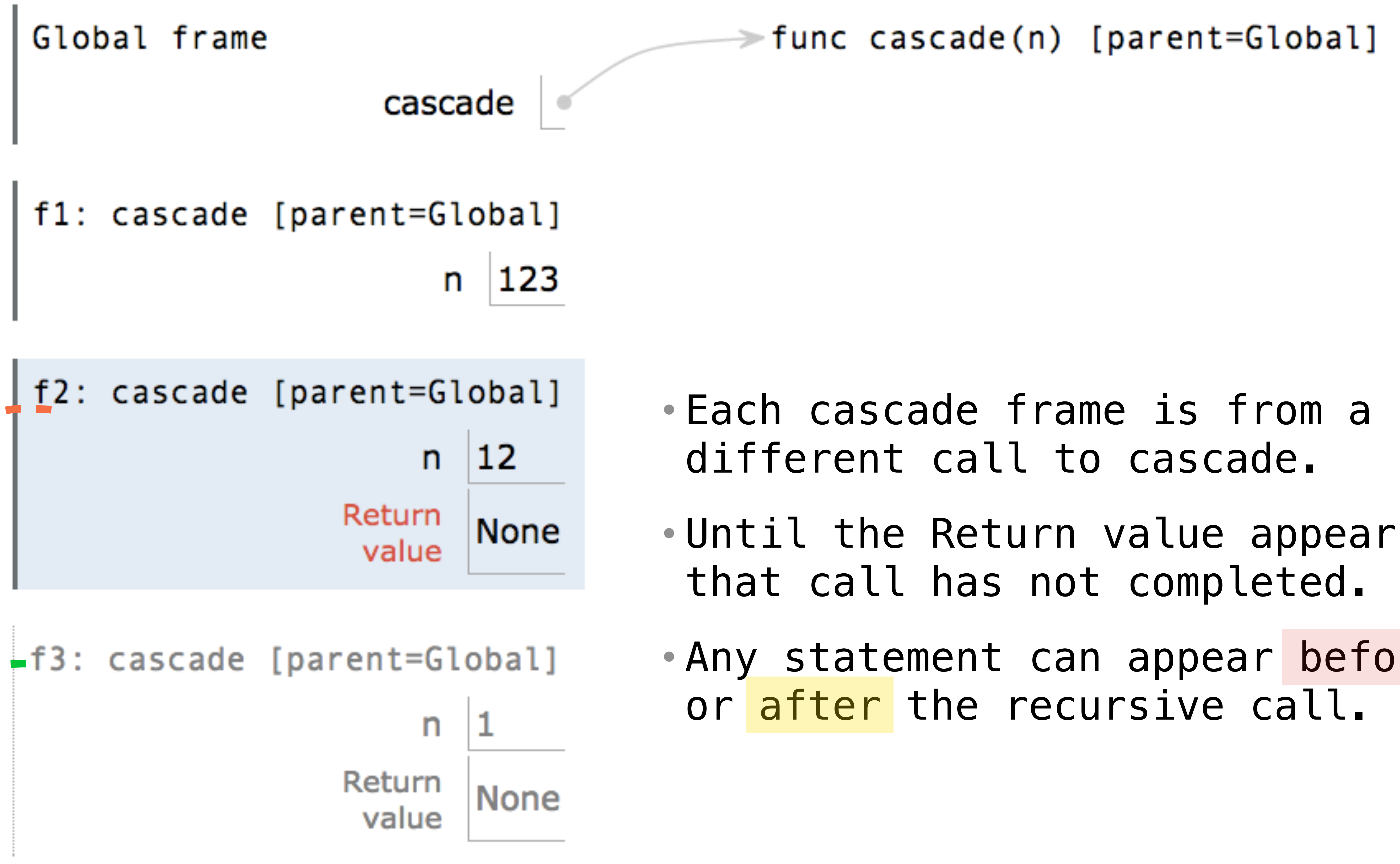
Order of Recursive Calls

The Cascade Function

(Demo)

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
```



Program output:

123
12
1
12

- Each cascade frame is from a different call to cascade.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.

Two Definitions of Cascade

(Demo)

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

```python
def cascade(n):
    print(n)
    if n >= 10:
        cascade(n//10)
        print(n)
```

- If two implementations are equally clear, then shorter is usually better
- In this case, the longer implementation is more clear (at least to me)
- When learning to write recursive functions, put the base cases first
- Both are recursive functions, even though only the first has typical structure

Example: Inverse Cascade

Inverse Cascade

Write a function that prints an inverse cascade:

```
1
12
123
1234
123
12
1
```

```
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow =   lambda n: f_then_g(                    )
shrink = lambda n: f_then_g(                    )
```