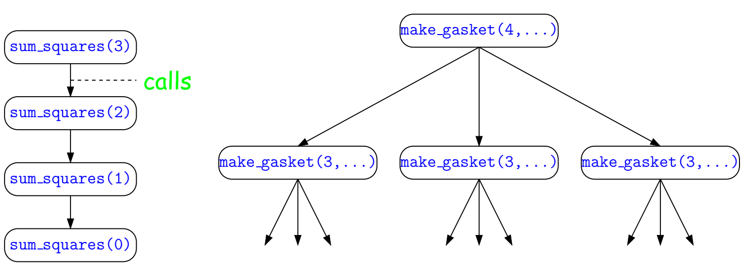
Lecture #7: Tree Recursion

Announcements

- Hog Contest and Hog Dice Design released today! Exercise your strategy- and artistic-design skills on the Game of Hog.
- Please fill out our Week 3 survey (Piazza note @500) to help us adjust the course effectively.
- There have been questions about what Python features one may use to complete the Hog project (among other things). Generally, you can get points for passing the tests by any means on the Python version used by the autograder. However, you may lose composition points as a result of straying into features we haven't gotten to yet.
- You can sign up for the Berkeley Programming Contest on 11 February. We use this to choose teams for the ACM International Collegiate Programming Contest, the first round of which is in March. Next week's contest will be entirely online, and will use the North American Qualifier contest. See Piazza post @536 for details and signup link.
- Ask questions on the Piazza thread for today's lecture (@575).

Tree Recursion

- The `make_gasket` function is an example of a *tree recursion*, where each call makes multiple recursive calls on itself.
- A *linear recursion* makes at most one recursive call per call.
- A *tail recursion* has at most one recursive call per call, and it is the last thing evaluated.
- A linear recursion such as for `sum_squares` produces the pattern of calls on the left, while `make_gasket` produces the pattern on the right—an instance of what we call a *tree* in computer science.



A Problem

Try to implement the following:

```
def find_zero(lowest, highest, func):
    """Return a value v such that LOWEST <= v <= HIGHEST and
    FUNC(v) == 0, or None if there is no such value.
    Assumes that FUNC is a non-decreasing function from integers
    to integers (that is, if a < b, then FUNC(a) <= FUNC(b)."""

    if ??:
        return None
    ??
```

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```
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    """Return a value v such that LOWEST <= v <= HIGHEST and
    FUNC(v) == 0, or None if there is no such value.
    Assumes that FUNC is a non-decreasing function from integers
    to integers (that is, if a < b, then FUNC(a) <= FUNC(b)."""

    if lowest > highest:     # Base Case
         return None
    elif ??:
         return lowest
    ??
```


A Problem

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```
def find_zero(lowest, highest, func):
    """Return a value v such that LOWEST <= v <= HIGHEST and
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    to integers (that is, if a < b, then FUNC(a) <= FUNC(b)."""

    if lowest > highest:     # Base Case
         return None
    elif func(lowest) == 0:
        return lowest        # Base Case
    else:
        ??
```


A Problem

Try to implement the following:

```
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    """Return a value v such that LOWEST <= v <= HIGHEST and
    FUNC(v) == 0, or None if there is no such value.
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    to integers (that is, if a < b, then FUNC(a) <= FUNC(b)."""

    if lowest > highest:     # Base Case
         return None
    elif func(lowest) == 0:  # Base Case
        return lowest
    else:                    # Inductive (Recursive) Case
        return find_zero(lowest + 1, highest, func)
```


A Problem

Try to implement the following:

```
def find_zero(lowest, highest, func):
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What kind of recursion is this?

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```

What kind of recursion is this?

Tail Recursion

```
# Equivalent iterative solution
while lowest <= highest:
    if func(lowest) == 0:
        return lowest
    lowest += 1
# If we get here, returns None
```


Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:
         return None
    ??
```


Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:
         return None
    middle = (lowest + highest) // 2
    if func(middle) == 0:   # Guess is correct
        return middle
    ??
```


Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:
         return None
    middle = (lowest + highest) // 2
    if func(middle) == 0:
        return middle
    elif func(middle) < 0:  # Guess is too low, result must be > middle
        return ??
    ??
```

Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:
         return None
    middle = (lowest + highest) // 2
    if func(middle) == 0:
        return middle
    elif func(middle) < 0:
        return find_zero(middle + 1, highest, func)
    else:                   # Guess is too high, result must be < middle
        return ??
```


Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:    # Base Case
         return None
    middle = (lowest + highest) // 2
    if func(middle) == 0:   # Base Case
        return middle
    elif func(middle) < 0:  # Inductive Case
        return find_zero(middle + 1, highest, func)
    else:                   # Inductive Case
        return find_zero(lowest, middle - 1, func)
```

What kind of recursion is this?

Problem, Take 2

Can make it faster by using the fact that the function is non-decreasing.

```
def find_zero(lowest, highest, func):
    ...
    if lowest > highest:    # Base Case
         return None
    middle = (lowest + highest) // 2
    if func(middle) == 0:   # Base Case
        return middle
    elif func(middle) < 0:  # Inductive Case
        return find_zero(middle + 1, highest, func)
    else:                   # Inductive Case
        return find_zero(lowest, middle - 1, func)
```

What kind of recursion is this?

Tail Recursion:

Two calls, but only one executed.

```
# Equivalent iterative solution
while lowest <= highest:
    middle = (lowest + highest) // 2
    if func(middle) == 0:
        return middle
    elif func(middle) < 0:
        lowest = middle + 1
    else:
        highest = middle - 1
```


Side Trip: Base Cases Without If

Can you do this without an **if** statement (just **and**/**or**)?

```
def is_a_zero(lowest, highest, func):
    """Return true iff there is a value v such that LOWEST <= v <= HIGHEST
    and FUNC(v) == 0.  Assumes that FUNC is a non-decreasing function
    from integers to integers."""

    middle = (lowest + highest) // 2

    return ??
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    from integers to integers."""

    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (??)
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    from integers to integers."""

    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (func(middle) == 0 \
                or ??)
```


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    from integers to integers."""

    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (func(middle) == 0 \
                or (func(middle) < 0 and is_a_zero(middle + 1, highest, func))
                or (func(middle) > 0 and is_a_zero(lowest, middle - 1, func)))
```


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    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (func(middle) == 0 \
                or (func(middle) < 0 and is_a_zero(middle + 1, highest, func))
                or (func(middle) > 0 and is_a_zero(lowest, middle - 1, func)))
```

What kind of recursion is this? Linear Recursion

Only one of the two calls to `is_a_zero` can happen, but if the first one evaluates to `False`, we still have to evaluate `func(middle)>0`. Thus the recursive call is *not* the last thing executed.

Side Trip: Base Cases Without If

Can you do this without an **if** statement (just **and/or**)?

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    """Return true iff there is a value v such that LOWEST <= v <= HIGHEST
    and FUNC(v) == 0.  Assumes that FUNC is a non-decreasing function
    from integers to integers."""

    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (func(middle) == 0 \
                or (func(middle) < 0 and is_a_zero(middle + 1, highest, func))
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    from integers to integers."""

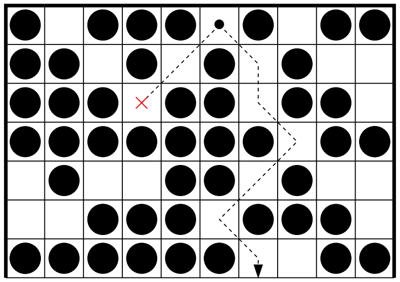
    middle = (lowest + highest) // 2

    return lowest <= highest \
           and (func(middle) == 0 \
                or (func(middle) < 0 and is_a_zero(middle + 1, highest, func))
                or is_a_zero(lowest, middle - 1, func))
```

What kind of recursion is this? Tree Recursion

Finding a Path

- Consider the problem of finding your way through a maze of blocks:

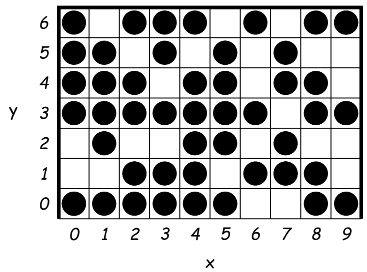


- From a given starting square, one can move down one row and up to one column left or right on each step, as long as the square moved to is unoccupied.
- Problem is to find a path to the bottom layer.
- Diagram shows one path that runs into a dead end (X) and one that escapes.

Path-Finding Program

- Translating the problem into a function specification:

```
def is_path(blocked, x0, y0):
    """True iff there is a path of squares from (X0, Y0) to some
    square (x1, 0) such that all squares on the path (including first and
    last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
    is true iff the grid square at (x, y) is occupied or off the edge.
    Each step of a path goes down one row and 1 or 0 columns left or right."""
```



This grid would be represented by a predicate M where, e.g, `M(0,0), M(1,0), M(1,2), not M(1, 1), not M(2,2).`

Here, `is_path(M, 5, 6)` is true;
`is_path(M, 1, 6)` and `is_path(M, 6, 6)` are false.

is_path Solution (I)

```
def is_path(blocked, x0, y0):
    """True iff there is a path of squares from (X0, Y0) to some
    square (x1, 0) such that all squares on the path (including first and
    last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
    is true iff the grid square at (x, y) is occupied or off the edge.
    Each step of a path goes down one row and 1 or 0 columns left or right."""

    if ____________:

        return ____________

    elif ____________:

        return ____________
    else:

        return __________________________________________________
```


is_path Solution (II)

```
def is_path(blocked, x0, y0):
    """True iff there is a path of squares from (X0, Y0) to some
    square (x1, 0) such that all squares on the path (including first and
    last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
    is true iff the grid square at (x, y) is occupied or off the edge.
    Each step of a path goes down one row and 1 or 0 columns left or right."""

    if ____________:

        return False

    elif ____________:

        return True
    else:

        return __________________________________________________
```


is_path Solution (III)

```
def is_path(blocked, x0, y0):
    """True iff there is a path of squares from (X0, Y0) to some
    square (x1, 0) such that all squares on the path (including first and
    last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
    is true iff the grid square at (x, y) is occupied or off the edge.
    Each step of a path goes down one row and 1 or 0 columns left or right."""

    if blocked(x0, y0):

        return False

    elif ____________:

        return True
    else:

        return __________________________________________________
```

is_path Solution (IV)

```
def is_path(blocked, x0, y0):
   """True iff there is a path of squares from (X0, Y0) to some
   square (x1, 0) such that all squares on the path (including first and
   last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge.
   Each step of a path goes down one row and 1 or 0 columns left or right."""

   if blocked(x0, y0):

       return False

   elif y0 == 0:

       return True
   else:

       return ________________________________________________
```


is_path Solution (V)

```
def is_path(blocked, x0, y0):
   """True iff there is a path of squares from (X0, Y0) to some
   square (x1, 0) such that all squares on the path (including first and
   last) are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge.
   Each step of a path goes down one row and 1 or 0 columns left or right."""

   if blocked(x0, y0):

       return False

   elif y0 == 0:

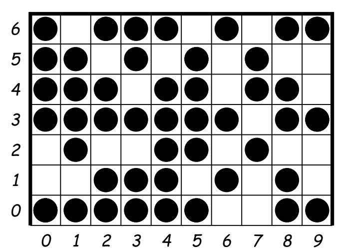
       return True
   else:

       return is_path(blocked, x0-1, y0-1)  \
              or is_path(blocked, x0, y0-1)  \
              or is_path(blocked, x0+1, y0-1)
```


Counting the Paths

```
def num_paths(blocked, x0, y0):
   """Return the number of unoccupied paths that run from (X0, Y0)
   to some square (x1, 0). BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge. """
```

For the previous predicate `M`, the result of `num_paths(M, 5, 6)` is 1.
For the predicate `M2`, denoting this grid (missing (7, 1)):



the result of `num_paths(M2, 5, 6)` is 5.

num_paths Solution (I)

```
def num_paths(blocked, x0, y0):
   """Return the number of unoccupied paths that run from (X0, Y0)
   to some square (x1, 0). BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge. """

   if blocked(x0, y0):

       return ________________________

   elif y0 == 0:

       return ________________________
   else:

       return ________________________________________________
```

num_paths Solution (II)

```
def num_paths(blocked, x0, y0):
   """Return the number of unoccupied paths that run from (X0, Y0)
   to some square (x1, 0). BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge. """

   if blocked(x0, y0):

       return 0

   elif y0 == 0:

       return 1
   else:

       return ____________________________________________
```


num_paths Solution (III)

```
def num_paths(blocked, x0, y0):
   """Return the number of unoccupied paths that run from (X0, Y0)
   to some square (x1, 0). BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied or off the edge. """

   if blocked(x0, y0):

       return 0

   elif y0 == 0:

       return 1
   else:

       return num_paths(blocked, x0-1, y0-1) \
            + num_paths(blocked, x0, y0-1) \
            + num_paths(blocked, x0+1, y0-1)
```


A Change in Problem

- Suppose we changed the definition of "path" for the maze problems to allow paths to go left or right without going down.
- And suppose we changed solutions in the obvious way, so that instead of just having recursive calls for the three squares

  (x_0-1, y_0-1), (x_0, y_0-1), and (x_0-1, y_0+1),

  we added calls for the two other squares

  (x_0-1, y_0) and (x_0+1, y_0).
- Will this work? What would happen?

A Change in Problem

- Suppose we changed the definition of "path" for the maze problems to allow paths to go left or right without going down.
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  (x_0-1, y_0-1), (x_0, y_0-1), and (x_0-1, y_0+1),

  we added calls for the two other squares

  (x_0-1, y_0) and (x_0+1, y_0).
- Will this work? What would happen?

  Infinite recursions, such as

  $$(8,2) \rightarrow (9,2) \rightarrow (8,2) \rightarrow \cdots$$

And a Little Analysis

- All our linear recursions took time proportional (in some sense) to the size of the problem.
- What about `is_path`?

And a Little Analysis

- All our linear recursions took time proportional (in some sense) to the size of the problem.
- What about `is_path`?

  Each call can spawn three others, for up to y0 "generations." That means the number of possible calls could be as many as `3 ** y0`—exponential growth.

Another Recursion Problem: Counting Partitions

- I'd like to know the number of distinct ways of expressing an integer as a sum of positive integer "parts."
- To make things more interesting, let's also limit the size of the integer parts to some given value:

```
def num_partitions(n, k):
    """Returns number of distinct ways to express N as a sum of positive
    integers each of which is <= K, where K > 0. (Empty sum is 0.)"""
```

- Example:

$$\left.\begin{aligned}6 &= 3+3\\ &= 3+2+1\\ &= 3+1+1+1\\ &= 2+2+2\\ &= 2+2+1+1\\ &= 2+1+1+1+1\\ &= 1+1+1+1+1+1\end{aligned}\right\}\text{Each line is one partition}$$

so `num_partitions(6, 3)` is 7.

Identifying the Problem in the Problem

- Again, consider `num_partitions(6, 3)`.
- Some partitions will contain the maximum size integer, 3, and the rest won't.
- Those that do contain 3 then have various ways to partition the remaining 3.

```
3     +     3
3     +     2     + 1
3     +     1 + 1 + 1
```

- While those that do not contain 3 partition 6 using integers no larger than 2:

```
2   +   2   +   2
2   +   2   +   1 + 1
2   +   1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1
```

- These observations generalize, and lead immediately to a solution.

Counting Partitions: Code (I)

```
def num_partitions(n, k):
    """Number of distinct ways to express N as a sum of positive
    integers each of which is <= K, where K > 0.  (The empty sum is 0.)"""

    if ________________________:

        return 0

    elif ________________________:

        return 1

    else:

        return ________________________________:
```


Counting Partitions: Code (II)

```
def num_partitions(n, k):
    """Number of distinct ways to express N as a sum of positive
    integers each of which is <= K, where K > 0.  (The empty sum is 0.)"""

    if n < 0:
        return 0

    elif ________________________:

        return 1

    else:

        return ________________________________:
```


Counting Partitions: Code (III)

```
def num_partitions(n, k):
    """Number of distinct ways to express N as a sum of positive
    integers each of which is <= K, where K > 0.  (The empty sum is 0.)"""

    if n < 0:
        return 0

    elif k == 1:

        return 1

    else:

        return ________________________________:
```


Counting Partitions: Code (IV)

```
def num_partitions(n, k):
    """Number of distinct ways to express N as a sum of positive
    integers each of which is <= K, where K > 0.  (The empty sum is 0.)"""

    if n < 0:
        return 0

    elif k == 1:

        return 1

    else:

        return num_partitions(n - k, k) + num_partitions(n, k - 1)
```

Recurrences

- The partition problem is a typical example of a mathematical *recurrence relation*.
- A familiar oneis the *Fibonacci sequence*, defined by

$$\text{fib}(n) = \begin{cases} 1, & \text{if } n \in \{0, 1\} \\ \text{fib}(n-2) + \text{fib}(n-1), & \text{if } n > 1 \end{cases}$$

- Which of course translates immediately to:

```
def fib(n):
    if n == 0 or n == 1:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```

- Giving us the sequence (for increasing values of n)

  0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

- Again, this is a tree recursion requiring an exponential amount of computation.
- But as we will see later, both here and in all the examples we've seen so far, dramatic speedup is possible.