

# Recursion

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## Announcements

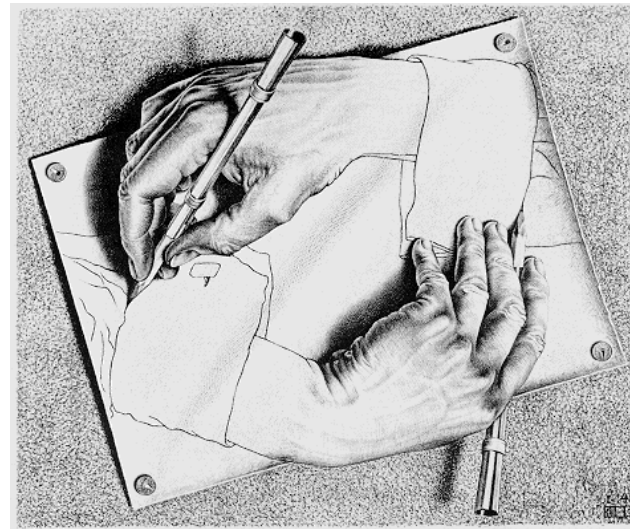
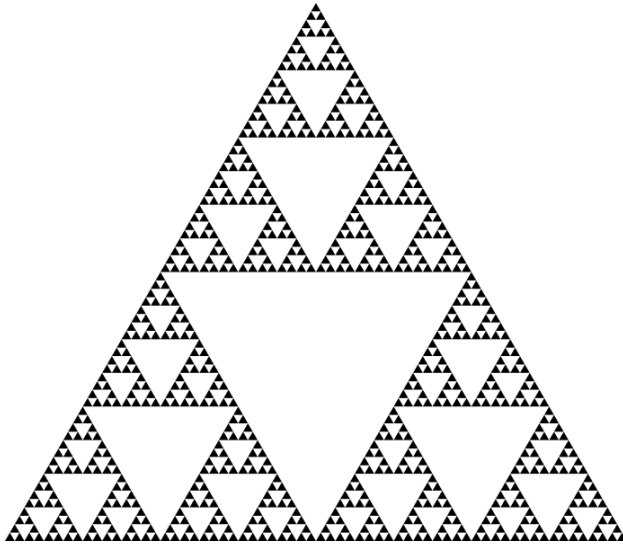
# Recursive Functions

## Recursive Functions

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**Definition:** A function is called recursive if the body of that function calls itself, either directly or indirectly

**Implication:** Executing the body of a recursive function may require applying that function



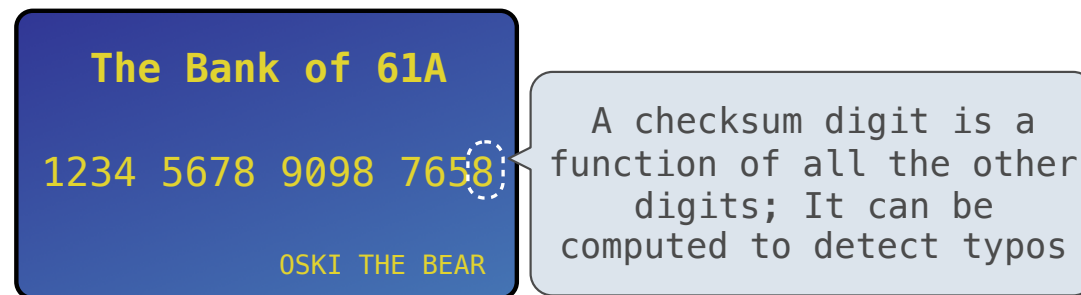
Drawing Hands, by M. C. Escher (lithograph, 1948)

## Digit Sums

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$$2+0+1+9 = 12$$

- If a number  $a$  is divisible by 9, then `sum_digits(a)` is also divisible by 9
- Useful for typo detection!



- Credit cards actually use the Luhn algorithm, which we'll implement after `sum_digits`

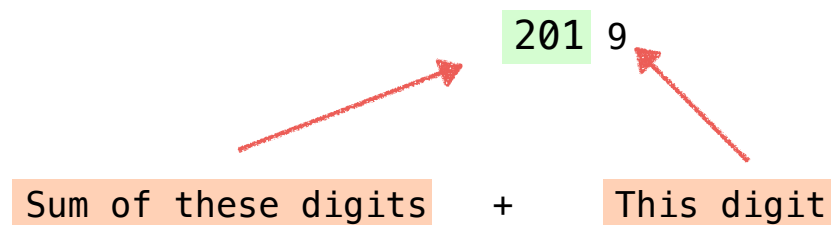
## The Problem Within the Problem

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The sum of the digits of 6 is 6.

Likewise for any one-digit (non-negative) number (i.e.,  $< 10$ ).

The sum of the digits of 2019 is



That is, we can break the problem of summing the digits of 2019 into a [smaller instance of the same problem](#), plus some extra stuff.

We call this [recursion](#)

## Sum Digits Without a While Statement

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```
def split(n):  
    """Split positive n into all but its last digit and its last digit."""  
    return n // 10, n % 10  
  
def sum_digits(n):  
    """Return the sum of the digits of positive integer n."""  
    if n < 10:  
        return n  
    else:  
        all_but_last, last = split(n)  
        return sum_digits(all_but_last) + last
```

## The Anatomy of a Recursive Function

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- The `def` statement header is similar to other functions
- Conditional statements check for `base cases`
- Base cases are evaluated `without recursive calls`
- Recursive cases are evaluated `with recursive calls`

```
def sum_digits(n):  
    """Return the sum of the digits of positive integer n."""  
    if n < 10:  
        return n  
    else:  
        all_but_last, last = split(n)  
        return sum_digits(all_but_last) + last
```

(Demo)



## Recursion in Environment Diagrams



## Iteration vs Recursion

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Iteration is a special case of recursion

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Using while:

```
def fact_iter(n):  
    total, k = 1, 1  
    while k <= n:  
        total, k = total*k, k+1  
    return total
```

Math:

$$n! = \prod_{k=1}^n k$$

Names:

n, total, k, fact\_iter

Using recursion:

```
def fact(n):  
    if n == 0:  
        return 1  
    else:  
        return n * fact(n-1)
```

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

n, fact

## Verifying Recursive Functions

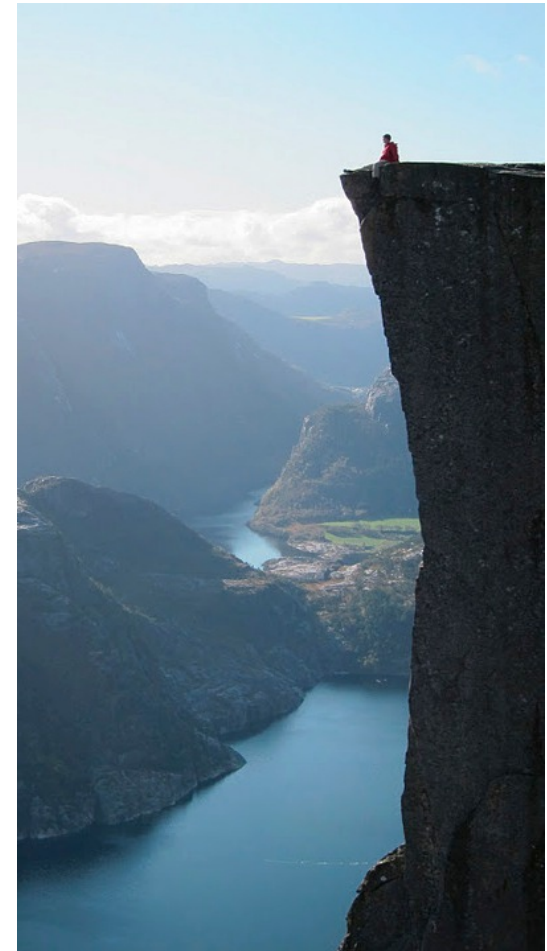
## The Recursive Leap of Faith

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```
def fact(n):  
    if n == 0:  
        return 1  
    else:  
        return n * fact(n-1)
```

Is fact implemented correctly?

1. Verify the base case
2. Treat `fact` as a functional abstraction!
3. Assume that `fact(n-1)` is correct
4. Verify that `fact(n)` is correct



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Photo by Kevin Lee, Preikestolen, Norway

## Mutual Recursion

## The Luhn Algorithm

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Used to verify credit card numbers

From Wikipedia: [http://en.wikipedia.org/wiki/Luhn\\_algorithm](http://en.wikipedia.org/wiki/Luhn_algorithm)

- **First:** From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g.,  $7 * 2 = 14$ ), then sum the digits of the products (e.g., 10:  $1 + 0 = 1$ , 14:  $1 + 4 = 5$ )
- **Second:** Take the sum of all the digits

1	3	8	7	4	3
2	3	1+6=7	7	8	3

 = 30

The Luhn sum of a valid credit card number is a multiple of 10

(Demo)

# Recursion and Iteration



## Converting Recursion to Iteration

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Can be tricky: Iteration is a special case of recursion.

Idea: Figure out what state must be maintained by the iterative function.

```
def sum_digits(n):  
    """Return the sum of the digits of positive integer n."""  
    if n < 10:  
        return n  
    else:  
        all_but_last, last = split(n)  
        return sum_digits(all_but_last) + last
```

What's left to sum

A partial sum

(Demo)

## Converting Iteration to Recursion

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More formulaic: Iteration is a special case of recursion.

Idea: The state of an iteration can be passed as arguments.

```
def sum_digits_iter(n):  
    digit_sum = 0  
    while n > 0:  
        n, last = split(n)  
        digit_sum = digit_sum + last  
    return digit_sum
```

Updates via assignment become...

```
def sum_digits_rec(n, digit_sum):  
    if n == 0:  
        return digit_sum  
    else:  
        n, last = split(n)  
        return sum_digits_rec(n, digit_sum + last)
```

...arguments to a recursive call