Tree Recursion
Class outline:

- Order of recursive calls
- Tree recursion
- Counting partitions
Order of recursive calls
The cascade function

def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)

- Each cascade frame is from a different call to cascade.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.
Print output:

```
123
12
1
12
123
```
Two definitions of cascade

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

```python
def cascade(n):
    print(n)
    if n >= 10:
        cascade(n//10)
        print(n)
```

- If two implementations are equally clear, then the shorter one is usually better
- When learning to write recursive functions, put the base cases first
- Both are recursive functions, even though only the first has typical structure
Inverse cascade

How can we output this cascade instead?

1
12
123
12
1
Inverse cascade solution

def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

grow = lambda n: f_then_g(n)
shrink = lambda n: f_then_g(n)

View in PythonTutor
def inverse_cascade(n):
grow(n)
print(n)
shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(
shrink = lambda n: f_then_g(

View in PythonTutor
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g( )

View in PythonTutor
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g(print, shrink, n//10)

View in PythonTutor
Tree recursion
(Multiple recursion)
Tree Recursion

Tree-shaped processes arise whenever a recursive function makes more than one recursive call.

Sierpinski curve
Recursive Virahanka-Fibonacci

The $n$th number is defined as:

$$
virfib(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
virfib(n - 1) + virfib(n - 2) & \text{otherwise}
\end{cases}
$$

```python
def virfib(n):
    """Compute the $n$th Virahanka-Fibonacci number, for $N \geq 1$."
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
```
Recursive Virahanka-Fibonacci

The $n$th number is defined as:

$$
virfib(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
virfib(n - 1) + virfib(n - 2) & \text{otherwise}
\end{cases}
$$

```python
def virfib(n):
    """Compute the $n$th Virahanka-Fibonacci number, for $N \geq 1$."
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return virfib(n-1) + virfib(n-2)
```
A tree-recursive call graph
Redundant computations

The function is called on the same number multiple times. 😞
(We will speed up this computation dramatically in a few weeks by remembering results)
Counting partitions
Counting partitions problem

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

```ruby
count_partitions(6, 4) # n, m
```

$$
\begin{align*}
2 + 4 &= 6 \\
1 + 1 + 4 &= 6 \\
3 + 3 &= 6 \\
1 + 2 + 3 &= 6 \\
1 + 1 + 1 + 3 &= 6 \\
2 + 2 + 2 &= 6 \\
1 + 1 + 2 + 2 &= 6 \\
1 + 1 + 1 + 1 + 2 &= 6 \\
1 + 1 + 1 + 1 + 1 + 1 &= 6 
\end{align*}
$$
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[ \text{count_partitions}(6, 4) \ # \ n, m \]

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\texttt{count_partitions(6, 4) # n, m}

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

\texttt{Use at least one 4}
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

```python
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

- Use at least one 4
- Don't use any 4
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

```
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

- Use at least one 4
- Don't use any 4

Tree recursion often involves exploring different choices.
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count_partitions}(6, 4) \# n, m
\]

Solve two simpler problems:

\[
\text{count_partitions}(2, 4)
\]
\[
\text{count_partitions}(6, 3)
\]
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count_partitions}(6, 4) \quad \# \quad n, m
\]

Solve two simpler problems:

\[
\begin{align*}
\text{count_partitions}(2, 4) \\
\text{count_partitions}(n-m, m)
\end{align*}
\]

\[
\text{count_partitions}(6, 3)
\]
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

$count\_partitions(6, 4) \# n, m$

Solve two simpler problems:

$count\_partitions(2, 4)$
$count\_partitions(n-m, m)$

$count\_partitions(6, 3)$
$count\_partitions(n, m-1)$
Counting partitions code

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

count_partitions(6, 4) # n, m

Solve two simpler problems:

with parts of size $m$:

count_partitions(2, 4)
count_partitions(n-m, m)

without parts of size $m$:

count_partitions(6, 3)
count_partitions(n, m-1)

def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
Counting partitions code

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count_partitions}(6, 4) \quad \# \quad n, m
\]

Solve two simpler problems:

**with** parts of size \( m \):

\[
\text{count_partitions}(2, 4) \\
\text{count_partitions}(n-m, m)
\]

**without** parts of size \( m \):

\[
\text{count_partitions}(6, 3) \\
\text{count_partitions}(n, m-1)
\]

```python
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
```
else:
    with_m = count_partitions(n-m, m)
    without_m = count_partitions(n, m-1)
    return with_m + without_m
Counting partitions code

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

```python
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4) # n, m
    count_partitions(6, 3)
    count_partitions(n, m-1)
```
if n == 0:
    return 1
elif n < 0:
    return 0
elif m == 0:
    return 0
else:
    with_m = count_partitions(n-m, m)
    without_m = count_partitions(n, m-1)
    return with_m + without_m
Count partitions call graph
Count partitions variant

To save on unneeded calls, we can cap $m$, the maximum partition size, based on $n$. We can also add a base case for $m$ of size 1.

```python
def count_partitions(n, m):
    if m == 1 or n == 0:
        return 1
    elif n < 0:
        return 0
    else:
        # Number of partitions using a partition of size M
        leftover = n - m
        with_m = count_partitions(leftover, min(leftover, m))
        # Number of partitions using size up to M-1
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```
Variant call graph
Variant visualization
Variant visualization

(2, 2)

(6, 4)

(6, 3)
Variant visualization

(2, 2)

(0, 0)

(2, 1)
Variant visualization
Variant visualization

(3, 3)

(0, 0)

(3, 2)
Variant visualization

(3, 2)

(1, 1)

(3, 1)
Variant visualization
Variant visualization

(4, 2)

(2, 2)

(4, 1)
Variant visualization

(2, 2)

(0, 0)

(2, 1)