Tree Recursion
Class outline:

- Order of recursive calls
- Tree recursion
- Counting partitions
Order of recursive calls
The cascade function

def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
The cascade function

def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)

View in PythonTutor

- Each cascade frame is from a different call to cascade.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.
Print output:

```
123
12
12
12
123
```
Two definitions of cascade

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

```python
def cascade(n):
    print(n)
    if n >= 10:
        cascade(n//10)
        print(n)
```

- If two implementations are equally clear, then the shorter one is usually better
- When learning to write recursive functions, put the base cases first
- Both are recursive functions, even though only the first has typical structure
Inverse cascade

How can we output this cascade instead?

1
12
123
123
12
1
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

grow = lambda n: f_then_g(  
shrink = lambda n: f_then_g(  

View in PythonTutor
Inverse cascade solution

```python
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(
shrink = lambda n: f_then_g(
```

View in PythonTutor
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g(  

View in PythonTutor
Inverse cascade solution

def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)

grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g(print, shrink, n//10)

View in PythonTutor
Tree recursion
(Multiple recursion)
Tree Recursion

Tree-shaped processes arise whenever a recursive function makes more than one recursive call.

Sierpinski curve
Recursive Virahanka-Fibonacci

The $n$th number is defined as:

$$\text{virfib}(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
\text{virfib}(n - 1) + \text{virfib}(n - 2) & \text{otherwise}
\end{cases}$$

```python
def virfib(n):
    """Compute the $n$th Virahanka-Fibonacci number, for $N \geq 1$."
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
```
Recursive Virahanka-Fibonacci

The nth number is defined as:

\[
\text{virfib}(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
\text{virfib}(n - 1) + \text{virfib}(n - 2) & \text{otherwise}
\end{cases}
\]

def virfib(n):
    
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return virfib(n-1) + virfib(n-2)
A tree-recursive call graph
Redundant computations

The function is called on the same number multiple times. 😞
(We will speed up this computation dramatically in a few weeks by remembering results)
Counting partitions
Counting partitions problem

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count_partitions}(6, 4) \# n, m
\]

- \( 2 + 4 = 6 \)
- \( 1 + 1 + 4 = 6 \)
- \( 3 + 3 = 6 \)
- \( 1 + 2 + 3 = 6 \)
- \( 1 + 1 + 1 + 3 = 6 \)
- \( 2 + 2 + 2 = 6 \)
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Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count_partitions}(6, 4) \# n, m
\]

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

```
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

Use at least one 4
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

```
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

- Use at least one 4
- Don't use any 4
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

```python
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

**Use at least one 4**

**Don't use any 4**

Tree recursion often involves exploring different choices.
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

$count_{\text{partitions}}(6, 4) \# n, m$

Solve two simpler problems:

$count_{\text{partitions}}(2, 4)$

$count_{\text{partitions}}(6, 3)$
Counting partitions approach

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

\[
\text{count_partitions}(6, 4) \# n, m
\]

Solve two simpler problems:

\[
\text{count_partitions}(2, 4) \\
\text{count_partitions}(n-m, m)
\]

\[
\text{count_partitions}(6, 3)
\]
Counting partitions approach

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

\[
\text{count}_\text{partitions}(6, 4) \# n, m
\]

Solve two simpler problems:

\[
\text{count}_\text{partitions}(2, 4) \\
\text{count}_\text{partitions}(n-m, m)
\]

\[
\text{count}_\text{partitions}(6, 3) \\
\text{count}_\text{partitions}(n, m-1)
\]
Counting partitions code

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

```python
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

**with** parts of size \( m \):

```python
count_partitions(2, 4)
count_partitions(n-m, m)
```

**without** parts of size \( m \):

```python
count_partitions(6, 3)
count_partitions(n, m-1)
```

```python
def count_partitions(n, m):
    
    def
    >>> count_partitions(6, 4)
```
Counting partitions code

The number of partitions of a positive integer \( n \), using parts up to size \( m \), is the number of ways in which \( n \) can be expressed as the sum of positive integer parts up to \( m \) in increasing order.

```python
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

**with** parts of size \( m \):

```python
count_partitions(2, 4)
count_partitions(n-m, m)
```

**without** parts of size \( m \):

```python
count_partitions(6, 3)
count_partitions(n, m-1)
```

```python
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
    ```
else:
    with_m = count_partitions(n-m, m)
    without_m = count_partitions(n, m-1)
    return with_m + without_m
Counting partitions code

The number of partitions of a positive integer $n$, using parts up to size $m$, is the number of ways in which $n$ can be expressed as the sum of positive integer parts up to $m$ in increasing order.

count_partitions(6, 4) # n, m

Solve two simpler problems:

with parts of size $m$:

count_partitions(2, 4)
count_partitions(n-m, m)

without parts of size $m$:

count_partitions(6, 3)
count_partitions(n, m-1)

def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
```python
if n == 0:
    return 1
elif n < 0:
    return 0
elif m == 0:
    return 0
else:
    with_m = count_partitions(n-m, m)
    without_m = count_partitions(n, m-1)
    return with_m + without_m
```
Count partitions call graph
Count partitions variant

To save on unneeded calls, we can cap $m$, the maximum partition size, based on $n$. We can also add a base case for $m$ of size 1.

```python
def count_partitions(n, m):
    if m == 1 or n == 0:
        return 1
    elif n < 0:
        return 0
    else:
        # Number of partitions using a partition of size M
        leftover = n - m
        with_m = count_partitions(leftover, min(leftover, m))
        # Number of partitions using size up to M-1
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```
Variant call graph
Variant visualization
## Variant visualization

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Variant visualization

(2, 2)
Variant visualization
Variant visualization
Variant visualization

(3, 2)

(1, 1)

(3, 1)
Variant visualization
Variant visualization
Variant visualization

(2, 2)

(0, 0)

(2, 1)