Lecture #17: Complexity and Orders of Growth

Complexity

- Certain problems take longer than others to solve, or require more storage space to hold intermediate results.
- We refer to the *time complexity* or *space complexity* of a problem.
- But what does it mean to say that a certain *program* has a particular complexity?
- What does it mean for an *algorithm*?
- What does it mean for a *problem*?

A Direct Approach

- Well, if you want to know how fast something is, you can time it.
- Python happens to make this easy:

```
>>> def fib(n):
...     if n <= 1: return n
...     else: return fib(n-2) + fib(n-1)
...
>>> from timeit import repeat
>>> repeat('fib(10)', globals=globals(), number=5)
[0.00029..., 0.00027..., 0.00027...]
>>> repeat('fib(20)', globals=globals(), number=5)
[0.021..., 0.017..., 0.017...]
>>> repeat('fib(30)', globals=globals(), number=5)
[2.26..., 2.25..., 2.25...]
```

- `repeat(Stmt, globals=globals(), number=N)` means

  Execute Stmt (a string) N times. Repeat this process 3 times and report the time required for each repetition. The 'globals' argument here makes sure that 'fib' is available.

A Direct Approach, Continued

- You can also use this from the command line (assuming that 'fib' is defined in file fib.py):

```
$ python3 -m timeit --setup='from fib import fib' 'fib(10)'
10000 loops, best of 3: 97 usec per loop
```

- This command automatically chooses a number of executions of fib to give a total time that is large enough for an accurate average, repeats 3 times, and reports the best time.

Strengths and Problems with Direct Approach

- Good: Gives actual times; answers question completely for given input and machine.
- Bad: Results apply only to tested inputs.
- Bad: Results apply only to particular programs, platforms, and loads.
- Bad: Cannot tell us anything about complexity of algorithm or of problem.

But Can't We Extrapolate?

- Why not try a succession of times, and use that to figure out timing in general? Here's an example using the Unix shell:

```
$ for t in 5 10 15 20 25 30; do
>     echo -n "$t: "
>     python3 -m timeit --setup='from fib import fib' "fib($t)"
> done
5: 100000 loops, best of 3: 2.04 usec per loop
10: 10000 loops, best of 3: 22.5 usec per loop
15: 1000 loops, best of 3: 256 usec per loop
20: 100 loops, best of 3: 2.75 msec per loop
25: 10 loops, best of 3: 30.6 msec per loop
30: 10 loops, best of 3: 345 msec per loop
```

- This is (very roughly) $1.5\,t^{1.6}$ usec when $t \geq 10$.
- But it still applies to a particular program and machine, and only looks at a few possible input values.

Worst Case, Best Case, Average Case

- To avoid the problem of getting results only for particular inputs, we usually ask a more general question, such as:
  - What is the *worst case* time to compute f(X) as a function of the size of X? or
  - what is the *average case* time to compute f(X) as a function of the size of X? or
  - what is the *best case* time to compute f(X) as a function of the size of X? or
- Here, "size" depends on the problem: could be magnitude of numeric input, number of digits, length (of list), cardinality (of set), etc.
- Average case can be hard and best case uninteresting. Here, we'll mostly be interested in worst cases.
- But now we seem to have a harder problem than before: how do we get worst-case times? Doesn't that require testing all cases?
- And when we do, aren't we still sensitive to machine model, compiler, etc.?

Example: Linear Search

- Consider the following search function:

```
def near(L, x, delta):
    """True iff X differs from some member of sequence L by no
    more than DELTA."""
    for y in L:
        if abs(x-y) <= delta:
            return True
    return False
```

- There's a lot here we don't know:
  - How long is sequence `L`?
  - Where in `L` is `x` (if it is)?
  - How long does it take to compare numbers `L`?
  - How long do `abs` and subtract take?
  - How long does it take to create an iterator for `L` and how long does its `__next__` operation take?
- So what can we meaningfully say about complexity of `near`?

What to Measure?

- If we want general answers, we have to introduce some "strategic vagueness."
- Instead of looking at times, we can consider number of "operations." Which?
- The total time consists of

  1. Some fixed overhead to start the function and begin the loop.
  2. Per-iteration costs: subtraction, `abs`, `__next__`, <=
  3. Some cost to end the loop.
  4. Some cost to return.

- So we can collect total operations into one "fixed-cost operation" (items 1, 3, 4), plus $M(L)$ "loop operations" (item 2), where $M(L)$ is the number of items in `L` up to and including the `y` that comes within `delta` of `x` (or the length of `L` if no match).

What Does an "Operation" Cost?

- But these "operations" are of different kinds and complexities, so what do we really know?
- Assuming that each operation represents some range of possible minimum and maximum values (constants), we can say that

$$\begin{aligned} & \textit{min-fixed-cost} + M(\mathsf{L}) \times \textit{min-loop-cost} \\ \leq\; & \\ & C_{\text{near}}(L) \\ \leq\; & \\ & \textit{max-fixed-cost} + M(\mathsf{L}) \times \textit{max-loop-cost} \end{aligned}$$

  where $C_{\text{near}}(L)$ is the cost of `near` on a list where the program has to look at $M(L)$ items.
- In the worst case $M(L) = ??$ and in the best, $M(L) = ??$, so

  $\textit{min-fixed-cost} \leq C_{\text{near}}(L) \leq \textit{max-fixed-cost} + \mathsf{len(L)} \times \textit{max-loop-cost}$.
- Simpler, but still clumsy, and the numbers are not going to be precise anyway. Would be nice to have a cleaner notation.

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- In the worst case $M(L) = \mathsf{len}(L)$ and in the best, $M(L) = ??$, so

  min-fixed-cost $\leq C_{\rm near}(L) \leq$ *max-fixed-cost*$+$len(L)$\times$*max-loop-cost*.

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- In the worst case $M(L) = \text{len}(L)$ and in the best, $M(L) = 0$, so

  $\textit{min-fixed-cost} \leq C_{\text{near}}(L) \leq \textit{max-fixed-cost} + \text{len(L)} \times \textit{max-loop-cost}$.

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Operation Counts and Scaling

- Instead of getting precise answers in units of physical time, we therefore settle for a proxy measure that will remain meaningful over changes in architecture or compiler.
- Choose some operations of interest and count how many times they occur.
- Examples:
  - How many times does fib get called recursively during computation of fib(N)?
  - How many addition operations get performed by fib(N)?
- You can no longer get precise times, but if the operations are well-chosen, results are *proportional* to actual time for different values of N.
- Thus, we look at how computation time *scales* in the worst case.
- Can compare programs/algorithms on the basis of which scale better.

Asymptotic Results

- Sometimes, results for "small" values are not indicative.
- E.g., suppose we have a prime-number tester that contains a look-up table of the primes up to 1,000,000,000 (about 50 million primes).
- Tests for numbers up to 1 billion will be faster than for larger numbers.
- So in general, we tend to ask about *asymptotic* behavior of programs: as size of input goes to infinity.

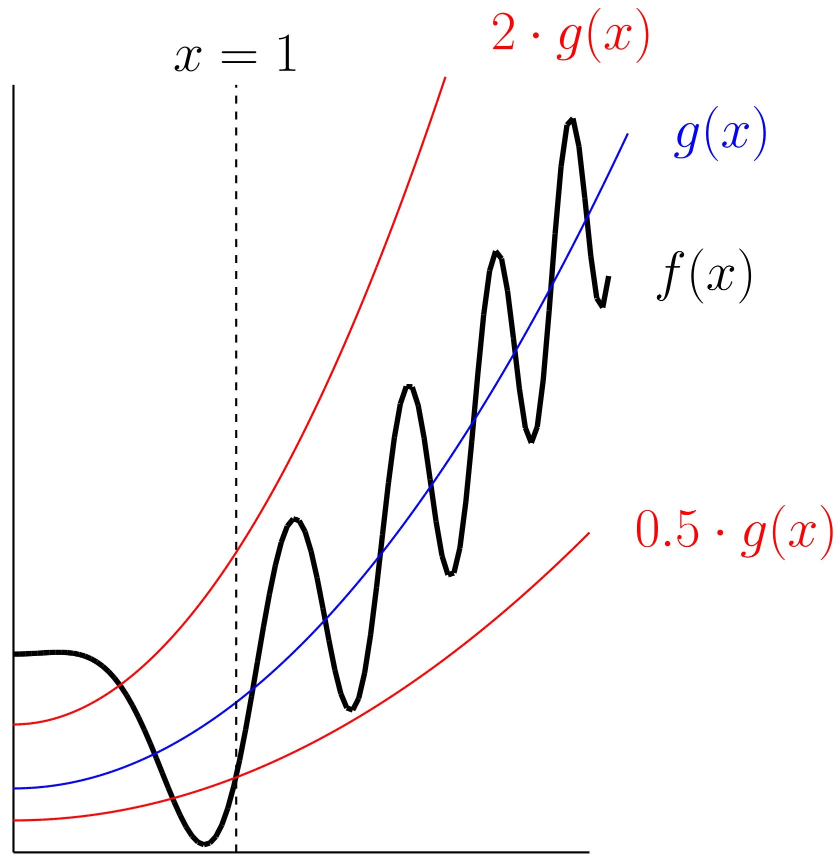
Expressing Approximation

- So, we are looking for measures of program performance that *give* us a sense of how computation time scales with size of input.
- And we are further interested in ignoring finite sets of special cases that a *given* program can compute quickly.
- Finally, precise worst-case functions can be very complicated, and the precision is generally not terribly important anyway.
- These considerations motivate the use of *order notation* to express how approximations of execution time or space grow.

The Notation

- Suppose that $f(n)$ and $g(n)$ are functions returning real numbers.
- For us, $f(n)$ will generally be some kind of cost function—the execution time for a problem of size n.
- We use the notation $\Theta(g(n))$ to mean "the set of all functions whose absolute values are *eventually* proportional to $g(n)$.
- We write $f(n) \in \Theta(g(n))$ to mean "whenever n is large enough, $K_1|g(n)| \leq |f(n)| \leq K_2|g(n)|$ where $0 < K_1 < K_2$ are some constants."
- In other words "$|f(n)|$ is roughly proportional to $|g(n)|$."
- This notation can be used to express the growth rate of any function, but again, we're mostly interested in cost functions.
- (BTW: in most other places, you'll see this written as $f(n) = \Theta(g(n))$, but I consider that nonsense, since $f(n)$ is a function and $\Theta(g(n))$ is a set of functions.)

Illustration



- Here, $f(x) \in \Theta(g(x))$ because once x is large enough ($x > 1$), $|f(x)|$ is always between two multiples of $|g(x)|$: $0.5 \cdot |g(x)| \leq |f(x)| \leq 2 \cdot |g(x)|$.

Using Asymptotic Estimates (I)

- Going back to the `near` function,

$$\begin{aligned} & \textit{min-fixed-cost} + M(L) \times \textit{min-loop-cost} \\ \leq\ & C_{\rm near}(L) \\ \leq\ & \textit{max-fixed-cost} + M(L) \times \textit{max-loop-cost} \end{aligned}$$

  where $M(L)$ is the number of items in L that are examined before the loop terminates.

- *In the worst case,* $M(L) = N$, where N is the length of L.

- So, letting $C^{\rm wc}_{\rm near}(N)$ mean "the worst-case value of $C_{\rm near}(L)$ when N is the length of L":

$$\begin{aligned} & \textit{min-fixed-cost} + N \times \textit{min-loop-cost} \\ \leq\ & C^{\rm wc}_{\rm near}(N) \\ \leq\ & \textit{max-fixed-cost} + N \times \textit{max-loop-cost} \end{aligned}$$

Using Asymptotic Estimates (II)

- We can state

$$\begin{aligned} & \textit{min-fixed-cost} + N \times \textit{min-loop-cost} \\ \le\ & C^{\text{wc}}_{\text{near}}(N) \\ \le\ & \textit{max-fixed-cost} + N \times \textit{max-loop-cost} \end{aligned}$$

  more cleanly as

$$C^{\text{wc}}_{\text{near}}(N) \in \Theta(N).$$

- Why?
- Well, if we ignore the two fixed costs (assume they are 0), we obviously fit the definition, since for $N \ge 0$,

$$p \cdot N \le C^{\text{wc}}_{\text{near}}(N) \le q \cdot N,$$

  where p is min-loop-cost and q is max-loop-cost (both constants).
- It's easy to see that we can arrange that when $N > 1$, we cover the necessary range by tweaking q up a bit—e.g., to

$$q + \textit{max-fixed-cost}$$

Typical $\Theta(\cdot)$ Estimates from Programs

Bound on Worst-Case Time	Example
Constant time $\Theta(1)$	`x += L[c]`
Logarithmic time $\Theta(\lg N)$	`while N > 0:` `    x, N = x + L[N], N // 2`
Linear time $\Theta(N)$	`for c in range(N):` `    x += L[c]`

Typical $\Theta(\cdot)$ Estimates from Programs (II)

<table>
<tr><th>Bound on Worst-Case Time</th><th>Example</th></tr>
<tr><td>$\Theta(N \lg N)$</td><td><pre>
def sort(L):   # Define N = len(L)
    M = len(L) // 2
    if M == 0: return L # Assume merge takes Θ(N)
    else: return merge(sort(L[:M]), sort(L[M:]))
</pre></td></tr>
<tr><td>Quadratic time
$\Theta(N^2)$</td><td><pre>
for c in range(N):   # Executed N times.
    for d in range(N):   # Executed N times for each c
        x += L[c][d]   # Executed N x N times.
</pre></td></tr>
<tr><td>Exponential time
$\Theta(2^N)$</td><td><pre>
def longMax(A, L, U):   # Define N = U-L; L<=U
    if L == U: return A[L]
    else: return max(longMax(A, L+1, U),
                     longMax(A, L, U-1))
</pre></td></tr>
</table>

Some Intuition on Meaning of Growth

- How big a problem can you solve in a *given* time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size N, assuming perfect scaling and that problem size 1 takes 1μsec.
- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.

Time (μsec) for problem size N	**Max N Possible in** 1 second	1 hour	1 month	1 century
$\lg N$	10^{300000}	$10^{1000000000}$	$10^{8\cdot 10^{11}}$	$10^{9\cdot 10^{14}}$
N	10^6	$3.6 \cdot 10^9$	$2.7 \cdot 10^{12}$	$3.2 \cdot 10^{15}$
$N \lg N$	63000	$1.3 \cdot 10^8$	$7.4 \cdot 10^{10}$	$6.9 \cdot 10^{13}$
N^2	1000	60000	$1.6 \cdot 10^6$	$5.6 \cdot 10^7$
N^3	100	1500	14000	150000
2^N	20	32	41	51

Efficiency and Complexity

- The term *efficiency* is often misused to describe what I've been discussing here.
- Efficiency is slightly different, however. We can define the efficiency of my program on a problem instance P as

$$\frac{\textit{theoretically optimal}\text{ time to compute result of } P}{\text{execution time of my program on } P}$$

- In the absence of a known theoretical result, we can use the performance of some "good" program.
- But this does raise the question: How does one determine theoretical optimums?

Lower Bounds on Algorithms

- A result that says "No algorithm for problem P can possibly have an asymptotic complexity of less than $\Theta(f(n))$" is called a *lower-bound result.*
- These are really hard to prove. You are basically saying "No matter how smart you are or how far technology advances, you'll never do better than this bound."
- A few are easy. For example, to add two integers, you cannot do any better than $\Theta(N)$, where N is the total number of digits (why?).
- Many are unsolved. For example, if you can prove that $P \neq NP$, you will win several prestigious prizes and $$.
- (If you prove that instead $P = NP$, you could bring about the end of civilization as we know it, but that's another story.)