Lecture #21: Complexity, Memoization
How Fast Is This (I)?

- For this program (L is a list and N <= len(L)):
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
  ```

- What is the worst-case time, measured in number of comparisons?

- What is the worst-case time, measured in number of additions (+=)?

- How about here?
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
          break
  ```
How Fast Is This (I)?

• For this program (L is a list and N ≤ len(L)):

```python
for x in range(N):
    # Answer: Θ(N) comparisons
    if L[x] < 0:
        c += 1
```

• What is the worst-case time, measured in number of comparisons?

• What is the worst-case time, measured in number of additions (+=)?

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for x in range(N):
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How Fast Is This (I)?

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        # Answer: Θ(N) comparisons
        c += 1
# Answer: Θ(N) additions
```

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• What is the worst-case time, measured in number of additions (+=)?

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for x in range(N):
    if L[x] < 0:
        c += 1
break
```
How Fast Is This (I)?

• For this program (L is a list and N <= len(L)):
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
  ```
  
  # Answer: $\Theta(N)$ comparisons
  # Answer: $\Theta(N)$ additions

• What is the worst-case time, measured in number of comparisons?
• What is the worst-case time, measured in number of additions (+=)?

• How about here?
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
          break
  ```
  
  # Answer: $\Theta(N)$ comparisons
 How Fast Is This (I)?

• For this program (L is a list and N <= len(L)):
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
  # Answer: Θ(N) comparisons
  # Answer: Θ(N) additions
  ```

• What is the worst-case time, measured in number of comparisons?

• What is the worst-case time, measured in number of additions (+=)?

• How about here?
  
  ```python
  for x in range(N):
      if L[x] < 0:
          c += 1
          break
  # Answer: Θ(N) comparisons
  # Answer: Θ(1) additions
  ```
How Fast Is This (II)?

- Assume that execution of \( f \) takes constant time.
- What is the complexity of this program, measured by number of calls to \( f \)? (Simplest answer)

```python
for x in range(2*N):
    f(x, x, x)
for y in range(3*N):
    f(x, y, y)
for z in range(4*N):
    f(x, y, z)
```
How Fast Is This (II)?

- Assume that execution of $f$ takes constant time.
- What is the complexity of this program, measured by number of calls to $f$? (Simplest answer)

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```

# Answer: $\Theta(N^3)$
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```

# Answer: $\Theta(N^3)$

• Why not $\Theta(24N^3 + 6N^2 + 2N)$?
How Fast Is This (II)?

• Assume that execution of $f$ takes constant time.

• What is the complexity of this program, measured by number of calls to $f$? (Simplest answer)

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for x in range(2*N):
    f(x, x, x)
for y in range(3*N):
    f(x, y, y)
for z in range(4*N):
    f(x, y, z)
```

# Answer: $\Theta(N^3)$

• Why not $\Theta(24N^3 + 6N^2 + 2N)$? That's correct, but equivalent to the simpler answer of $\Theta(N^3)$. 

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How Fast Is This (III)?

• What is the complexity of this program, measured by number of calls to \( f \)?

```python
for x in range(N):
    for y in range(x):
        f(x, y)
```
How Fast Is This (III)?

• What is the complexity of this program, measured by number of calls to $f$?

```python
for x in range(N):    # Answer $\Theta(N^2)$
    for y in range(x):
        f(x, y)
```

• The complexity is given by an arithmetic series:

$$0 + 1 + 2 + \cdots + N - 1 = N(N-1)/2 \in \Theta(N^2).$$

• Again, constant factors ($1/2$) and linear terms ($N/2$) are ignorable.
How Fast Is This (IV)?

- What about this one, measured by number of calls to $f$? (Careful! This is tricky.)

- How about measured by number of comparisons ($<$)?

\[
\begin{align*}
z &= 0 \\
\text{for } x \text{ in range}(N): \\
\quad \text{for } y \text{ in range}(N): \\
\quad \quad \text{while } z < N: \\
\quad \quad \quad f(x, y, z) \\
\quad \quad z &= z + 1
\end{align*}
\]
How Fast Is This (IV)?

• What about this one, measured by number of calls to \( f \)? (Careful! This is tricky.)

• How about measured by number of comparisons (<)?

\[
\begin{align*}
z &= 0 \\
for \ x \ in \ range(N): & \quad \# \text{Answer } \Theta(N) \text{ calls to } f. \\
\quad for \ y \ in \ range(N): & \\
\quad \quad while \ z < N: & \\
\quad \quad \quad f(x, y, z) & \\
\quad \quad z &= 1 \\
\end{align*}
\]
How Fast Is This (IV)?

• What about this one, measured by number of calls to $f$? (Careful! This is tricky.)

• How about measured by number of comparisons ($<$)?

```python
z = 0
for x in range(N):  # Answer $\Theta(N)$ calls to $f$.
    for y in range(N):  # Answer $\Theta(N^2)$ comparisons.
        while z < N:
            f(x, y, z)
            z += 1
```

• **In practice**, which measure (calls to $f$ or comparisons) would matter?
How Fast Is This (IV)?

- What about this one, measured by number of calls to \( f \)? (Careful! This is tricky.)
- How about measured by number of comparisons (\(<\))?

\[
z = 0 \\
\text{for } x \text{ in range}(N): \quad \# \text{ Answer } \Theta(N) \text{ calls to } f. \\
\quad \text{for } y \text{ in range}(N): \quad \# \text{ Answer } \Theta(N^2) \text{ comparisons.} \\
\quad \text{while } z < N: \\
\quad \quad f(x, y, z) \\
\quad z += 1
\]

- In practice, which measure (calls to \( f \) or comparisons) would matter?
- Depends on size of \( N \), actual cost of \( f \). For large enough \( N \), comparisons will matter more.
New Subject: Avoiding Redundant Computation

- Consider again the classic Fibonacci recursion:
  
  ```python
  def fib(n):
      if n <= 1:
          return n
      else:
          return fib(n-1) + fib(n-2)
  ```

- This is a tree recursion with a serious speed problem.

![Diagram of Fibonacci tree recursion]

- Redundant computations and therefore computing time grow exponentially.
Avoiding Redundant Computation (II)

- The usual iterative version of \texttt{fib} does not have this problem because it saves the results of the recursive calls (in effect) and reuses them.

- Each computation of a number in the sequence happens exactly once, so the computation is linear in \( n \) (if we count additions as constant-time operations).

\begin{verbatim}
def fib(n):
    if n <= 1:
        return n
    a = 0
    b = 1
    for k in range(2, n+1):
        a, b = b, a+b
    return b
\end{verbatim}
Change Counting

• Consider the problem of determining the number of ways to give change for some amount of money:

```python
def count_change(amount, coins = (50, 25, 10, 5, 1)):
    """Return the number of ways to make change for AMOUNT, where the coin denominations are given by COINS."
    """
    if amount == 0:
        return 1
    elif len(coins) == 0 or amount < 0:
        return 0
    else:
        # = Ways with largest coin + Ways without largest coin
        return count_change(amount-coins[0], coins) + count_change(amount, coins[1:])
```

• Here, we often revisit the same subproblem:
  - E.g., Consider making change for 87 cents.
  - When we choose to use one half-dollar piece, we have the same subproblem (change for 37 cents) as when we choose to use no half-dollars and two quarters.
Memoizing

- Extending the iterative Fibonacci idea, let's keep around a table ("memo table") of previously computed values.
- Consult the table before using the full computation.
- **Example:** `count_change`:

```python
def count_change(amount, coins = (50, 25, 10, 5, 1)):
    memo_table = {}
    def count_change(amount, coins):
        key = (amount, coins)
        if key not in memo_table:
            memo_table[key] = full_count_change(amount, coins)
        return memo_table[key]
    def full_count_change(amount, coins):
        # original recursive solution goes here verbatim
        # when it calls count_change, calls memoized version.
        return count_change(amount,coins)
```

- **Question:** how could we test for infinite recursion?
Optimizing Memoization

- Used a dictionary to memoize `count_change`, which is highly general, but can be relatively slow.

- More often, we use arrays indexed by integers (lists in Python), but the idea is the same.

- For example, in the `count_change` program, we can index by `amount` and by the *number of coins* remaining in `coins`.

```python
def count_change(amount, coins = (50, 25, 10, 5, 1)):
    # memo_table[amt][k] contains the value computed for
    # count_change(amt, coins[k:])
    memo_table = [ [-1] * (len(coins)+1) for i in range(amount+1) ]

def count_change(amount, coins):
    if amount < 0: return 0
    elif memo_table[amount][len(coins)] == -1:
        memo_table[amount][len(coins)] = full_count_change(amount, coins)
        return memo_table[amount][len(coins)]

def full_count_change(amount, coins):
    # Full recursive version.
    return count_change(amount, coins)
```

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Order of Calls

• Going one step further, we can analyze the order in which our program ends up filling in the table.

• So consider adding some tracing to our memoized `count_change` program (using an extension of the `@trace1` decorator from Lecture #9.)

```python
memo_table = {}
def count_change(amount, coins):
    ... full_count_change(amount, coins) ...
    return memo_table[amount,coins]
@trace
def full_count_change(amount, coins):
    if amount == 0: return 1
    elif len(coins) == 0 or amount < 0: return 0
    else:
        return count_change(amount, coins[1:]) \
        + count_change(amount-coins[0], coins)
return count_change(amount,coins)
```
Result of Tracing

- **Consider** `count_change(57) (→ N means “returns N”)**:

  ```
  full_count_change(57, ()) -> 0  # Need shorter 'coins' arguments
  full_count_change(56, ()) -> 0  # first.
  ...
  full_count_change(1, ()) -> 0
  full_count_change(0, (1,)) -> 1  # For same coins, need smaller
  full_count_change(1, (1,)) -> 1  # amounts first.
  ...
  full_count_change(57, (1,)) -> 1
  full_count_change(2, (5, 1)) -> 1
  full_count_change(7, (5, 1)) -> 2
  ...
  full_count_change(57, (5, 1)) -> 12
  full_count_change(7, (10, 5, 1)) -> 2
  full_count_change(17, (10, 5, 1)) -> 6
  ...
  full_count_change(32, (10, 5, 1)) -> 16
  full_count_change(7, (25, 10, 5, 1)) -> 2
  full_count_change(32, (25, 10, 5, 1)) -> 2
  full_count_change(57, (25, 10, 5, 1)) -> 18
  full_count_change(57, (25, 10, 5, 1)) -> 60
  full_count_change(7, (50, 25, 10, 5, 1)) -> 2
  full_count_change(57, (50, 25, 10, 5, 1)) -> 62
  ```
Order of Calls (II)

- (New slide; not in lecture)
- We can see from the code that to compute the value of \texttt{full\_count\_change(N, C)}, it is sufficient to have
  - The values of \texttt{full\_count\_change(N, C[k:])} for $1 \leq k \leq \text{len}(C)$, and
  - The values of \texttt{full\_count\_change(k, C)} for $k < N$.
- And that tells us that, for example, we can compute all the values for \texttt{full\_count\_change(k, C)} for $C == ()$, then $C == (1,)$, then $C == (5, 1)$,
- And for each of these values of $C$, we can compute \texttt{full\_count\_change(k, C)} for all values of $k$ in order,
- ... and at each point, we will already have computed all the recursive call values we need.
## Filling in the Memo Table

<table>
<thead>
<tr>
<th>Amount</th>
<th>Coins Left</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1 2 3 4 5</td>
</tr>
<tr>
<td>0</td>
<td>1 1 1 1 1 1</td>
</tr>
<tr>
<td>1</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>2</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>3</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>4</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 2 2 2</td>
</tr>
<tr>
<td></td>
<td>Arrows show order of filling</td>
</tr>
<tr>
<td>23</td>
<td>0 1 5 9 9 9</td>
</tr>
<tr>
<td>24</td>
<td>0 1 5 9 9 9</td>
</tr>
<tr>
<td>25</td>
<td>0 1 6 12 13 13</td>
</tr>
<tr>
<td>26</td>
<td>0 1 6 12 13 13</td>
</tr>
<tr>
<td></td>
<td>Arrows show order of filling</td>
</tr>
<tr>
<td>54</td>
<td>0 1 11 36 49 50</td>
</tr>
<tr>
<td>55</td>
<td>0 1 12 42 60 62</td>
</tr>
<tr>
<td>56</td>
<td>0 1 12 42 60 62</td>
</tr>
<tr>
<td>57</td>
<td>0 1 12 42 60 62</td>
</tr>
</tbody>
</table>
Dynamic Programming

- Now rewrite `count_change` to make the order of calls explicit, so that we needn’t check to see if a value is memoized.

- Technique is called *dynamic programming* (for some reason).

- We start with the base cases (0 coins) and work backwards.

```python
def count_change(amount, coins = (50, 25, 10, 5, 1)):
    memo_table = [ [-1] * (len(coins)+1) for i in range(amount+1) ]
    def count_change(amount, coins):
        if amount < 0: return 0
        else: return memo_table[amount][len(coins)]
    def full_count_change(amount, coins):
        # How often called?
        ... # (calls count_change for recursive results)

    for a in range(0, amount+1):
        memo_table[a][0] = full_count_change(a, ())
    for k in range(1, len(coins) + 1):
        for a in range(1, amount+1):
            memo_table[a][k] = full_count_change(a, coins[-k:])
    return count_change(amount, coins)
```