Scopes & Tail Calls
Class outline:

- Lexical vs. dynamic scopes
- Recursion efficiency
- Tail recursive functions
- Tail call optimization
Scopes
Lexical scope

The standard way in which names are looked up in Scheme and Python.

Lexical (static) scope: The parent of a frame is the frame in which a procedure was defined

```
(define f (lambda (x) (+ x y)))
(define g (lambda (x y) (f (+ x x))))
(g 3 7)
```

Global frame

```
| f → λ (x) |
| g → λ (x, y) |
```

What happens when we run this code?

```
f1: g [parent=Global]
    | x | 3 |
    | y | 7 |
```

```
f2: f [parent=Global]
    | x | 6 |
```
Lexical scope

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**Lexical (static) scope:** The parent of a frame is the frame in which a procedure was defined

(define f (lambda (x) (+ x y)))
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Global frame

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>λ (x)</td>
<td>λ (x, y)</td>
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</table>

What happens when we run this code?
Error: unknown identifier: y

f1: g [parent=Global]

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>7</th>
</tr>
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<tr>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>y</td>
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f2: f [parent=Global]

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<tr>
<th></th>
<th>6</th>
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<td>x</td>
<td></td>
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Dynamic scope

An alternate approach to scoping supported by some languages.

**Dynamic scope**: The parent of a frame is the frame in which a procedure was called

Scheme includes the `mu` special form for dynamic scoping.

```
(define f (mu (x) (+ x y)))
(define g (lambda (x y) (f (+ x x))))
(g 3 7)
```

Global frame

```
g → λ (x, y)
```

What happens when we run this code?

```
     f → μ (x)
```

```
f1: g [parent=Global]
      x 3
      y 7
```

```
f2: f [parent=f1]
      x 6
```
Dynamic scope

An alternate approach to scoping supported by some languages.

**Dynamic scope:** The parent of a frame is the frame in which a procedure was called

Scheme includes the \texttt{mu} special form for dynamic scoping.

```
(define f (mu (x) (+ x y)))
(define g (lambda (x y) (f (+ x x))))
(g 3 7)
```

Global frame

\[
\begin{array}{c}
\text{Global frame} \\
\hline
f \rightarrow \mu (x) \\
g \rightarrow \lambda (x, y) \\
\end{array}
\]

What happens when we run this code?

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Recursion efficiency
# Recursion and iteration in Python

<table>
<thead>
<tr>
<th>Code</th>
<th>Time</th>
<th>Space</th>
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| ```python
def factorial(n, k):
    while n > 0:
        n = n - 1
        k = k * n
    return k
``` |      |       |
| ```python
def factorial(n, k):
    if n == 0:
        return k
    else:
        return factorial(n-1, k*n)
``` |      |       |
## Recursion and iteration in Python

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| ```python
def factorial(n, k):
    while n > 0:
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        k = k * n
    return k
``` | Linear |       |

```python
def factorial(n, k):
    if n == 0:
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```
Recursion and iteration in Python

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| def factorial(n, k):
  while n > 0:
    n = n - 1
    k = k * n
  return k | Linear | Constant |
| def factorial(n, k):
  if n == 0:
    return k
  else:
    return factorial(n-1, k*n) |       |        |
# Recursion and iteration in Python

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# Recursion and iteration in Python

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  return k | Linear | Constant |
| def factorial(n, k):
  if n == 0:
    return k
  else:
    return factorial(n-1, k*n) | Linear | Linear   |
Recursion frames in Python

In Python, recursive calls always create new frames.

def factorial(n, k):
    if n == 0:
        return k
    else:
        return factorial(n-1, k*n)

Active frames over time:

View in PythonTutor
Recursion in Scheme

In Scheme interpreters, a tail-recursive function should only require a **constant** number of active frames.

(define (factorial n k)
  (if (= n 0)
      k
      (factorial (- n 1) (* k n)))))

Active frames over time:

<table>
<thead>
<tr>
<th>Time</th>
<th>(fact 3 1)</th>
<th>(fact 2 3)</th>
<th>(fact 1 6)</th>
<th>(fact 0 6)</th>
<th>6</th>
</tr>
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Tail recursive functions
Tail recursive functions

In a **tail recursive function**, every recursive call must be a tail call.

```
(define (factorial n k)
  (if (= n 0)
      k
      (factorial (- n 1) (* k n)))
```

A **tail call** is a call expression in a **tail context**:

- The last body sub-expression in a **lambda** expression
- Sub-expressions 2 & 3 in a tail context **if** expression
- All non-predicate sub-expressions in a tail context **cond**
- The last sub-expression in a tail context **and**, **or**, **begin**, or **let**
Example: Length of list

(define (length s)
  (if (null? s) 0
  (+ 1 (length (cdr s)) )))

A call expression is not a tail call if more computation is still required in the calling procedure.

But linear recursive procedures can often be re-written to use tail calls...
Example: Length of list

```scheme
(define (length s)
  (if (null? s) 0
       (+ 1 (length (cdr s)) ) )
)
```

A call expression is not a tail call if more computation is still required in the calling procedure.

But linear recursive procedures can often be re-written to use tail calls...

```scheme
(define (length-tail s)
  (define (length-iter s n)
    (if (null? s) n
        (length-iter (cdr s) (+ 1 n)) )
  (length-iter s 0) )
)```
Is it tail recursive?

;; Compute the length of s.
(define (length s)
  (+ 1 (if (null? s)
         -1
         (length (cdr s)))))

;; Return whether s contains v.
(define (contains s v)
  (if (null? s)
      false
      (if (= v (car s))
          true
          (contains (cdr s) v))))
Is it tail recursive?

;;; Compute the length of s.
(define (length s)
  (+ 1 (if (null? s)
          -1
          (length (cdr s)))))

✗ No, because if is not in a tail context.

;;; Return whether s contains v.
(define (contains s v)
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Is it tail recursive?

;;; Compute the length of s.
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  (+ 1 (if (null? s)
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❌ No, because **if** is not in a tail context.

;;; Return whether s contains v.
(define (contains s v)
  (if (null? s)
      false
      (if (= v (car s))
         true
         (contains (cdr s) v)))))

✔ Yes, because **contains** is in a tail context **if**.
Is it tail recursive? 2

;;; Return whether s has any repeated elements.
(define (has-repeat s)
  (if (null? s)
      false
      (if (contains? (cdr s) (car s))
      true
      (has-repeat (cdr s)))))

;;; Return the nth Fibonacci number.
(define (fib n)
  (define (fib-iter current k)
      (if (= k n)
          current
          (fib-iter (+ current
            (fib (- k 1)))
            (+ k 1)) ))
  (if (= 1 n) 0 (fib-iter 1 2)))
Is it tail recursive? 2

```scheme
;; Return whether s has any repeated elements.
(define (has-repeat s)
  (if (null? s)
      false
      (if (contains? (cdr s) (car s))
          true
          (has-repeat (cdr s))))
)
```

✔ Yes, because `has-repeat` is in a tail context.

```scheme
;; Return the nth Fibonacci number.
(define (fib n)
  (define (fib-iter current k)
    (if (= k n)
        current
        (fib-iter (+ current
                    (fib (- k 1)))
                  (+ k 1))
    )
  )
  (if (= 1 n) 0 (fib-iter 1 2))
)```

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Is it tail recursive? 2

;;; Return whether s has any repeated elements.
(define (has-repeat s)
  (if (null? s)
      false
      (if (contains? (cdr s) (car s))
          true
          (has-repeat (cdr s)))) )

✔ Yes, because has-repeat is in a tail context.

;;; Return the nth Fibonacci number.
(define (fib n)
  (define (fib-iter current k)
    (if (= k n)
        current
        (fib-iter (+ current
                    (fib (- k 1)))
                   (+ k 1)) ) )
  (if (= 1 n) 0 (fib-iter 1 2)))

✗ No, because fib is not in a tail context.
Example: Reduce

(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
Example: Reduce

\[(\text{reduce } \ast \ '({3 \ 4 \ 5}) \ 2) \ 120\]
\[(\text{reduce } (\text{lambda } (x \ y) (\text{cons } y \ x)) \ '({3 \ 4 \ 5}) \ '({2}) \ ({5 \ 4 \ 3 \ 2})\]

\[(\text{define} \ (\text{reduce} \ \text{procedure} \ s \ \text{start})\]
\[
\quad (\text{if } \text{(null? } s) \ \text{start}\]
\quad (\text{reduce} \ \text{procedure}
\quad \quad (\text{cdr} \ s)
\quad \quad (\text{procedure} \ \text{start} \ (\text{car} \ s))) \ \)) \)

Is it tail recursive?
Example: Reduce

\[(\text{reduce } * \ (3 \ 4 \ 5) \ 2) \ 120\]
\[(\text{reduce } (\lambda x \ y \ (\text{cons } y x)) \ (3 \ 4 \ 5) \ (2)) \ (5 \ 4 \ 3 \ 2)\]

\[(\text{define } (\text{reduce } \text{procedure } s \ \text{start}) \]
\[\quad (\text{if } (\text{null? } s) \ \text{start} \]
\[\quad \quad (\text{reduce } \text{procedure} \]
\[\quad \quad \quad (\text{cdr } s) \]
\[\quad \quad \quad (\text{procedure } \text{start} \ (\text{car } s)) \ ) \ ) \ ) \ ]

Is it tail recursive?

✔ Yes, because \text{reduce} is in a tail context.
Example: Reduce

```
(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
```

```
(define (reduce procedure s start)
  (if (null? s) start
    (reduce procedure
      (reduce procedure
        (cdr s)
        (procedure start (car s))))))
```

Is it tail recursive?

✔ Yes, because `reduce` is in a tail context.

However, if `procedure` is not tail recursive, then this may still require more than constant space for execution.
Example: Map

```
(map (lambda (x) (- 5 x)) (list 1 2))
```
Example: Map

(map (lambda (x) (- 5 x)) (list 1 2))

(define (map procedure s)
  (if (null? s)
      nil
      (cons (procedure (car s))
            (map procedure (cdr s))))
)

Is it tail recursive?
Example: Map

(map (lambda (x) (- 5 x)) (list 1 2))

(define (map procedure s)
  (if (null? s)
      nil
      (cons (procedure (car s))
           (map procedure (cdr s)))))

Is it tail recursive?
✗ No, because map is not in a tail context.
Example: Map (Tail recursive)

```scheme
(define (map procedure s)
  (define (map-reverse s m)
    (if (null? s)
      m
      (map-reverse (cdr s) (cons (procedure (car s)) m))))
  (reverse (map-reverse s nil)))

(define (reverse s)
  (define (reverse-iter s r)
    (if (null? s)
      r
      (reverse-iter (cdr s) (cons (car s) r))))
  (reverse-iter s nil))

(map (lambda (x) (- 5 x)) (list 1 2))
```
Tail call optimization with trampolining
What the thunk?

**Thunk**: An expression wrapped in an argument-less function.

Making thunks in Python:

```python
thunk1 = lambda: 2 * (3 + 4)
thunk2 = lambda: add(2, 4)
```

Calling a thunk later:

```python
thunk1()
thunk2()
```
Trampolining

**Trampoline**: A loop that iteratively invokes thunk-returning functions.

```python
def trampoline(f, *args):
    v = f(*args)
    while callable(v):
        v = v()
    return v
```

The function needs to be thunk-returning! One possibility:

```python
def factorial_thunked(n, k):
    if n == 0:
        return k
    else:
        return lambda: factorial_thunked(n - 1, k * n)
```

```python
trampoline(factorial_thunked, 3, 1)
```

View in PythonTutor
Demo: Trampolined interpreter

The Scheme project EC is to implement trampolining. Let's see how it improves the ability to call tail recursive functions...