Tail Calls
Announcements
Dynamic Scope
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### Special form to create dynamically scoped procedures (mu special form only exists in Project 4 Scheme)

(mu

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13
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But... no **for/while** statements! Can we make basic iteration efficient? Yes!
Recursion and Iteration in Python

In Python, recursive calls always create new active frames

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\text{factorial}(n, k) \text{ computes: } n! \times k
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def factorial(n, k):
    if n == 0:
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Linear recursive procedures can often be re-written to use tail calls.
Example: Length of a List

(define (length s)
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(define (length-iter s n)
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(Demo)
Tail Recursion Examples
Which Procedures are Tail Recursive?

Which of the following procedures run in constant space?

;; Compute the length of s.
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;; Return whether s contains v.
(define (contains s v)
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;; Return whether s has any repeated elements.
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Map and Reduce
Example: Reduce
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(define (reduce procedure s start)
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(reduce * '(3 4 5) 2)
Example: Reduce

\[
\text{define (reduce procedure s start)}
\]

\[
\text{(reduce * '(3 4 5) 2)} \quad 120
\]
Example: Reduce

\[
\text{(define (reduce \text{procedure} \text{s} \text{start})}
\]

\[
\text{(reduce \ast \text{'}\text{(3 4 5) 2})}
\]

\[
\text{(reduce (lambda (x y) (cons y x)) \text{'}\text{(3 4 5) 2})}
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Example: Reduce

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(define (reduce procedure s start)

(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
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Example: Reduce

\[
\text{(define } \text{reduce procedure } s \text{ start)} \hfill \notag \\
\text{if (null? } s) \text{ start} \hfill \notag
\]

\[
(\text{reduce } \ast (3 \ 4 \ 5) \ 2) \quad 120 \hfill \notag \\
(\text{reduce } (\lambda (x \ y) (\text{cons } y \ x)) (3 \ 4 \ 5) \ '(2)) \quad (5 \ 4 \ 3 \ 2) \hfill \notag
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(define (reduce procedure s start)
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**Example: Reduce**

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\text{(define (reduce procedure s start)}\\
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\]

\[
\begin{align*}
\text{(reduce \ast \ '(3\ 4\ 5)\ 2)} & \quad 120 \\
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Recursive call is a tail call

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Recursive call is a tail call
Space depends on what procedure requires

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Example: Map with Only a Constant Number of Frames
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(define (map procedure s)
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(define (map procedure s)
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Example: Map with Only a Constant Number of Frames

(define (map procedure s)
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      ...))
Example: Map with Only a Constant Number of Frames

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\quad \quad \quad (\text{map} \ \text{procedure} \ (\text{cdr} \ s)))) \\
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\[
\text{(define (map procedure s)} \quad \text{(define (map-reverse s m)} \\
\text{\quad (if (null? s)} \quad \text{(define (map procedure s)} \\
\text{\quad \quad \quad nil)} \quad \text{(define (map-reverse s m)} \\
\text{\quad \quad \quad (cons (procedure (car s)}) \quad \text{(if (null? s)} \\
\text{\quad \quad \quad \quad (map procedure (cdr s)))))} \\
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\text{(map (lambda (x) (- 5 x)) (list 1 2))}
\]
Example: Map with Only a Constant Number of Frames

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(define (map procedure s)
  (if (null? s)
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```
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Example: Map with Only a Constant Number of Frames

\[
\begin{align*}
\text{(define (map procedure s)} & \text{)} \\
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\end{align*}
\]

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\text{(map (lambda (x) (- 5 x)) (list 1 2))} \\
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\begin{align*}
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& \end{align*}
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\]

\[
\text{(define (reverse s)}
\begin{align*}
& \text{ (define (reverse-iter s r)} \\
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```
General Computing Machines
An Analogy: Programs Define Machines
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Programs specify the logic of a computational device
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Programs specify the logic of a computational device

factorial
An Analogy: Programs Define Machines

Programs specify the logic of a computational device

\[
\text{factorial} = 1 
\]

\[
= 1 
\]

\[
\text{factorial} 
\]

\[
- 1 
\]

\[
\times 
\]

\[
= 
\]
An Analogy: Programs Define Machines

Programs specify the logic of a computational device

```
5 → factorial
    ↓ 1
    = 1
    ↓ 1
    ← 1
4 → factorial
    ↓ - factorial
    ← 1
3 → ...
```

`factorial` is a function that calculates the factorial of a number, starting from 5 and decreasing by 1 until 1, multiplying each result by the previous one until reaching 1.
An Analogy: Programs Define Machines

Programs specify the logic of a computational device

5 \rightarrow \text{factorial} \rightarrow - \rightarrow 1 \rightarrow = \rightarrow 1 \rightarrow \ast \rightarrow 1 \rightarrow 120
Interpreters are General Computing Machine
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An interpreter can be parameterized to simulate any machine
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

```
(define (factorial n)
  (if (zero? n) 1 (* n (factorial (- n 1)))))
```

5 → Scheme Interpreter → 120
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

\[
\text{(define (factorial n)} \\
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Our Scheme interpreter is a universal machine
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

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(define (factorial n)
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Our Scheme interpreter is a universal machine

A bridge between the data objects that are manipulated by our programming language and the programming language itself
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

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5 \rightarrow \text{Scheme Interpreter} \rightarrow 120
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\text{(define (factorial n)}
\text{(if (zero? n) 1 (* n (factorial (- n 1))))})
\]

Our Scheme interpreter is a universal machine

A bridge between the data objects that are manipulated by our programming language and the programming language itself

Internally, it is just a set of evaluation rules