Newton's Method

Newton's Method Background

Quickly finds accurate approximations to zeroes of differentiable functions!

- A "zero" of a function $f$ is an input $x$ such that $f(x) = 0$.

Application: a method for computing square roots, cube roots, etc.

The positive zero of $f(x) = x^2 - a$ is $\sqrt{a}$.

Newton's Method

Given a function $f$ and initial guess $x$,

Repeatedly improve $x$:
- Compute the value of $f$ at the guess: $f(x)$
- Compute the derivative of $f$ at the guess: $f'(x)$
- Update guess $x$ to be:
  \[ x = x - \frac{f(x)}{f'(x)} \]

Finish when $f(x) = 0$ (or close enough)

Using Newton's Method

How to find the square root of 27?

\[
\begin{align*}
\text{>>> } f &= \text{lambda } x: x^3 - 27 \\
\text{>>> } df &= \text{lambda } x: 3x^2 \\
\text{>>> } find_zero(f, df) &= 3
\end{align*}
\]

How to find the cube root of 729?

\[
\begin{align*}
\text{>>> } g &= \text{lambda } x: x^3 - 729 \\
\text{>>> } dg &= \text{lambda } x: 3x^2 \\
\text{>>> } find_zero(g, dg) &= 9
\end{align*}
\]

Special Case: Square Roots

How to compute $\text{square_root}(a)$

Idea: Iteratively refine a guess $x$ about the square root of $a$

Update: \[ x = \frac{x + a/x}{2} \]

Babylonian Method

Implementation questions:
- What guess should start the computation?
- How do we know when we are finished?
Special Case: Cube Roots

How to compute cube_root(a)

Idea: Iteratively refine a guess x about the cube root of a

Update: \( x = \frac{2 \cdot x + \frac{a}{x^2}}{3} \) (Demo)

Implementation questions:

What guess should start the computation?
How do we know when we are finished?

Implementing Newton's Method

Approximate Differentiation

Differentiation can be performed symbolically or numerically

\[
\begin{align*}
  f(x) &= x^2 - 16 \\
  f'(x) &= 2x \\
  f'(2) &= 4 \\
  f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
  f'(x) &= \frac{f(x + h) - f(x)}{h} \quad \text{(if } h \text{ is small)}
\end{align*}
\]

(Demo)

Critical Points and Inverses

Maxima, minima, and inflection points of a differentiable function occur when the derivative is 0

The inverse \( f^{-1}(y) \) of a differentiable, one-to-one function computes the value \( x \) such that \( f(x) = y \)

(Demo)