CS 61A/CS 98-52

Mehrdad Niknami

University of California, Berkeley
Today, we’re going to learn how to add & multiply. Exciting!

Let’s add two positive $n$-bit integers ($n = 8$ here):

Carry: 1 111111
Augend: 10110111
Addend: + 10011101
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Sum: 101010100

This is called ripple-carry addition.

Some questions:
1. How big can the sum be (at most)? What is the worst case?
2. How long does summation take in the worst case? Why?

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Mehrdad Niknami (UC Berkeley)
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Terminology

Some basic terminology:

Process: A running program

Threads: A unit of program flow (n threads = n independent executions of code)

Thread context: All the information a thread needs to run code, including the location of the code it is currently executing, as well as its current stack frame (local variables, etc.).

Concurrency: Overlapping operations (X begins before Y ends)

Parallelism: Simultaneously-occurring operations (multiple operations happening at the same time)
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- **Parallelism**: *Simultaneously-occurring operations* (multiple operations happening *at the same time*)
Parallel operations are always concurrent by definition. Concurrent operations need not be in parallel (e.g., open door, open window, close door, close window). Parallelism gives you a speed boost (multiple operations at the same time), but requires $N$ processors for $N \times$ speedup. Concurrency allows you to avoid stopping one thing before starting another, and can occur on a single processor.
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Distributed computation (running on multiple machines) is more difficult:

- Needs fault-tolerance (more machines = higher failure probability)
- Lack of shared memory
- More limited communication bandwidth (network slower than RAM)
- Time becomes problematic to handle

Rich literature, e.g. actor-based models of computation (MoC) such as discrete-event, synchronous-reactive, synchronous dataflow, etc. for analyzing/designing systems with guaranteed performance or reliability.
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Threading example:

```python
import threading
t = threading.Thread(target=print, args=('a',))
t.start()
print('b')  # may print 'b' before or after 'a'
t.join()  # wait for t to finish
```
Threading

Race condition:

```python
import threading
lst = [0]
def f():
    lst[0] += 1
    # write 1 might occur after read 2
    t = threading.Thread(target=f)
    t.start()
f()
t.join()
assert lst[0] in [1, 2]  # could be any of these!
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**Concurrency Control**

**Mutex (Lock in Python):** Object that can prevent concurrent access (mutual-exclusion).

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import threading
lock = threading.Lock()
lst = [0]

def f():
    lock.acquire()  # waits for mutex to be available
    lst[0] += 1  # only one thread may run this code
    lock.release()  # makes mutex available to others

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To obtain parallelism in CPython, you can use multiprocessing: running another copy of the program and communicating with it.

Jython, IronPython, etc. can run Python in parallel, and most other languages support parallelism as well.

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Threads/processes need to communicate. Common techniques:

- **Shared memory**: mutating shared objects (if all on 1 machine)
  - Pros: Reduces copying of data (faster/less memory)
  - Cons: Must block execution until lock is acquired (slow)

- **Message-passing**: sending data through thread-safe queues
  - Pros: Queue can buffer & work asynchronously (faster)
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- **Pipes**: synchronous version of message-passing ("rendezvous")
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Message-passing example for parallelizing $f(x) = x^2$:

```python
from multiprocessing import Process, Queue

def f(q_in, q_out):
    while True:
        x = q_in.get()
        if x is None:
            break
        q_out.put(x ** 2)  # real work

if __name__ == '__main__':  # only on main thread
    qs = (Queue(), Queue())
    procs = [Process(target=f, args=qs) for _ in range(4)]
    for proc in procs:
        proc.start()
    for i in range(10):
        qs[0].put(i)  # send inputs
    for i in range(10):
        print(qs[1].get())  # receive outputs
    for proc in procs:
        qs[0].put(None)  # notify finished
    for proc in procs:
        proc.join()
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Inter-Thread and Inter-Process Communication (IPC)

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Addition

Common parallelism technique: divide-and-conquer

1. Divide problem into separate subproblems
2. Solve subproblems in parallel
3. Merge sub-results into main result

XOR (and AND, and OR) are easy to parallelize:

1. Split each $n$-bit number into $p$ pieces
2. XOR each $n/p$-bit pair of numbers independently
3. Put back the bits together

Can we do something similar with addition?
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2. XOR each \( n/p \)-bit pair of numbers independently
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Can we do something similar with addition?
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Common parallelism technique: **divide-and-conquer**

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Let's go back to addition. We have two $n$-bit numbers to add. What if we take the same approach for $+$ as for XOR?

1. Split each $n$-bit number into $p$ pieces
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Key idea #1: A carry can be either 0 or 1... and we add different pieces in parallel... and then select the correct one based on carry! ⇒ This is called a carry-select adder.

Key idea #2: We can do this recursively. ⇒ This is called a conditional-sum adder.

How fast is a conditional-sum adder?
Running time is proportional to maximum propagation depth
We solve two problems of half the size simultaneously
We combine solutions with constant extra work
Therefore, parallel running time is $\Theta(\log n)$
However, we do more work:
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Other algorithms also exist with different trade-offs:

- Carry-skip adder
- Carry-lookahead adder (CLA)
- Kogge–Stone adder ("parallel-prefix" CLA; widely used)
- Brent-Kung adder
- Han–Carlson adder
- Lynch–Swartzlander spanning tree adder (fastest?)

...I don't know them. But $\Theta(\log n)$ is already asymptotically optimal. :-)

Some algorithms are better suited for hardware due to lower "fan-out":

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For two $n$-bit numbers, how long does it take in parallel?

Multiplication by 1 is a copy, taking $\Theta(1)$ depth.

There are $n$ additions.

Divide-and-conquer therefore takes $\Theta(\log n)$ additions.

Each addition takes $\Theta(\log n)$ depth.

Total depth is therefore $\Theta((\log n)^2)$.

...can we do better? :-)

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Carry-save addition:
reduce every $a + b + c$ into $r + s$ in parallel:

- Compute all carry bits $r$ independently
  - This is just OR, so $\Theta(1)$ depth
- Compute all sums-excluding-carries $s$ independently
  - This is just XOR, so $\Theta(1)$ depth
- Recurse on new $r_1 + s_1 + r_2 + s_2 + \ldots$ until final $r + s$ is obtained.
  - This takes $\Theta(\log n)$ levels of recursion
- Compute final sum in additional $\Theta(\log n)$ depth

Total depth is therefore $\Theta(\log n)$!

\[ ^2 \text{Simplified; detailed analysis is a little tedious. See here.} \]
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There isn't too much special about addition from basic arithmetic. Often the same tricks apply to any binary operator $\oplus$ that is associative!

Parallel addition can be generalized this way, called "parallel prefix":

Say we want to compute cumulative sum of 1, 2, 3, ...
First, group into binary tree: (((1 2) (3 4)) ((5 6) (7 8))) ...
Then, evaluate sums for all nodes recursively toward root
Finally, propagate sums back down from root to right-hand children

This is a very flexible operation, useful as a basic parallel building block.
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- Transformation assumed to ignore order (to allow parallelism)
Google recognized this and built a fast framework called MapReduce for automatically parallelizing & distributing such code across a cluster.

MapReduce: Simplified Data Processing on Large Clusters by Jeffrey Dean and Sanjay Ghemawat (2004)

System and method for efficient large-scale data processing

U.S. Patent 7,650,331

Fault-tolerance is handled automatically (why is this possible?)

Apache Hadoop later developed as an open-source implementation

"MapReduce" became a general programming model for distributed data processing

Spark (Matei Zaharia, UCB AMPLab, now at Databricks) developed as a faster implementation that processes data in RAM

Mehrdad Niknami (UC Berkeley)
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- **MapReduce: Simplified Data Processing on Large Clusters**
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*Spark* (Matei Zaharia, UCB AMPLab, now at Databricks) developed as a faster implementation that processes data in RAM
Parallel map is easy in Python!

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>>> from multiprocessing.pool import Pool
>>> pool = Pool()
>>> pool.map(math.sqrt, [1, 2, 3, 4])
[1.0, 1.4142135623730951, 1.7320508075688772, 2.0]
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