1 More Recursion

Questions

1.1 In discussion 1, we implemented the function `is_prime`, which takes in a positive integer and returns whether or not that integer is prime, iteratively.

Now, let’s implement it recursively! As a reminder, an integer is considered prime if it has exactly two unique factors: 1 and itself.

```python
def is_prime(n):
    ""
    >>> is_prime(7)
    True
    >>> is_prime(10)
    False
    >>> is_prime(1)
    False
    ""
    def prime_helper(______________):
        if ________________:
            ________________
        elif ________________:
            ________________
        else:
            ________________
        return ________________
```
def prime_helper(index):
    if index == n:
        return True
    elif n % index == 0 or n == 1:
        return False
    else:
        return prime_helper(index + 1)

return prime_helper(2)
1.2 Define a function `make_fn_repeater` which takes in a one-argument function `f` and an integer `x`. It should return another function which takes in one argument, another integer. This function returns the result of applying `f` to `x` this number of times.

Make sure to use recursion in your solution.

```python
def make_func_repeater(f, x):
    ""
    >>> incr_1 = make_func_repeater(lambda x: x + 1, 1)
    >>> incr_1(2)  # same as f(f(x))
    3
    >>> incr_1(5)
    6
    ""

    def repeat(___________________):
        if _________________:
            return ________________
        else:
            return ________________

    return ________________

def repeat(i):
    if i == 0:
        return x
    else:
        return f(repeat(i - 1))

return repeat
```
2 Tree Recursion

Consider a function that requires more than one recursive call. A simple example is the recursive fibonacci function:

```python
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

This type of recursion is called tree recursion, because it makes more than one recursive call in its recursive case. If we draw out the recursive calls, we see the recursive calls in the shape of an upside-down tree:

```
      fib(4)
    /      \
  fib(3)   fib(2)
  /      \
fib(2)   fib(1)  fib(1)  fib(0)
```

We could, in theory, use loops to write the same procedure. However, problems that are naturally solved using tree recursive procedures are generally difficult to write iteratively. It is sometimes the case that a tree recursive problem also involves iteration: for example, you might use a while loop to add together multiple recursive calls.

As a general rule of thumb, whenever you need to try multiple possibilities at the same time, you should consider using tree recursion.

How to diagram Tree Recursion

Questions

2.1 I want to go up a flight of stairs that has n steps. I can either take 1 or 2 steps each time. How many different ways can I go up this flight of stairs? Write a function count_stair_ways that solves this problem for me. Assume n is positive.

Before we start, what's the base case for this question? What is the simplest input?

When there is only 1 step, there is only one way to go up the stair. When there are two steps, we can go up in two ways: take a two-step, or take 2 one-steps.
What do \texttt{count\_stair\_ways(n - 1)} and \texttt{count\_stair\_ways(n - 2)} represent?

\texttt{count\_stair\_ways(n - 1)} represents the number of different ways to go up the first \(n - 1\) stairs. \texttt{count\_stair\_ways(n - 2)} represents the number of different ways to go up the first \(n - 2\) stairs. Our base cases will take care of the remaining 1 or 2 steps.

Use those two recursive calls to write the recursive case:

\begin{verbatim}
def count_stair_ways(n):
    if n == 1:
        return 1
    elif n == 2:
        return 2
    return count_stair_ways(n-1) + count_stair_ways(n-2)
\end{verbatim}

Video walkthrough (Leap of Faith)
Video Walkthrough (Diagramming Trees)

2.2 Consider a special version of the \texttt{count\_stairways} problem, where instead of taking 1 or 2 steps, we are able to take \textbf{up to and including} \(k\) steps at a time.

Write a function \texttt{count\_k} that figures out the number of paths for this scenario. Assume \(n\) and \(k\) are positive.

\begin{verbatim}
def count_k(n, k):
    """
    >>> count_k(3, 3) # 3, 2 + 1, 1 + 2, 1 + 1 + 1
    4
    >>> count_k(4, 4)
    8
    >>> count_k(10, 3)
    274
    >>> count_k(300, 1) # Only one step at a time
    1
    """
    if n == 0:
        return 1
    elif n < 0:
        return 0
    else:
        total = 0
\end{verbatim}
\[ i = 1 \]
\[ \text{while } i \leq k:\]
\[ \quad \text{total } += \text{count}_k(n - i, k) \]
\[ \quad i += 1 \]
\[ \text{return total} \]

Video Walkthrough
Here's a part of the Pascal’s triangle:

<table>
<thead>
<tr>
<th>Column:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 0:</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 1:</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 2:</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 3:</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 4:</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Every number in Pascal’s triangle is defined as the sum of the item above it and the item that is directly to the upper left of it, use 0 if the entry is empty. Define the procedure `pascal(row, column)` which takes a row and a column, and finds the value at that position in the triangle.

```python
def pascal(row, column):
    if column == 0:
        return 1
    elif row == 0:
        return 0
    else:
        return pascal(row - 1, column) + pascal(row - 1, column - 1)
```

**Background**: Pascal’s triangle is a useful recursive definition that tells us the coefficients in the expansion of the polynomial $(x + a)^n$. Each element in the triangle has a coordinate, given by the row it is on and its position in the row (which you could call its column).