1 Orders of Growth

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- **square(1)** requires one primitive operation: * (multiplication). **square(100)** also requires one. No matter what input $n$ we pass into **square**, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 · 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 · 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>square($n$)</td>
<td>$n$ · $n$</td>
<td>1</td>
</tr>
</tbody>
</table>

- **factorial(1)** requires one multiplication, but **factorial(100)** requires 100 multiplications. As we increase the input size of $n$, the runtime (number of operations) increases linearly proportional to the input.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 · 1 · 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 · 99 · · · · · · 1 · 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>factorial($n$)</td>
<td>$n$ · ($n$ − 1) · · · · · · 1 · 1</td>
<td>$n$</td>
</tr>
</tbody>
</table>

For expressing complexity, we use what is called big Θ (Theta) notation. For example, if we say the running time of a function foo is in $\Theta(n^2)$, we mean that the running time of the process will grow proportionally with the square of the size of the input as it increases to infinity.
• **Ignore lower order terms:** If a function requires $n^3 + 3n^2 + 5n + 10$ operations with a given input $n$, then the runtime of this function is $Θ(n^3)$. As $n$ gets larger, the lower order terms (10, $5n$, and $3n^2$) all become insignificant compared to $n^3$.

• **Ignore constants:** If a function requires $5n$ operations with a given input $n$, then the runtime of this function is $Θ(n)$. We are only concerned with how the runtime grows asymptotically with the input, and since $5n$ is still asymptotically linear; the constant factor does not make a difference in runtime analysis.

### Kinds of Growth

Here are some common orders of growth, ranked from no growth to fastest growth:

- $Θ(1)$ — constant time takes the same amount of time regardless of input size
- $Θ(\log n)$ — logarithmic time
- $Θ(n)$ — linear time
- $Θ(n \log n)$ — linearithmic time
- $Θ(n^2), Θ(n^3)$, etc. — polynomial time
- $Θ(2^n), Θ(3^n)$, etc. — exponential time (considered “intractable”; these are really, really horrible)

In addition, some programs will never terminate if they get stuck in an infinite loop.
Questions

What is the order of growth for the following functions?

1.1 def sum_of_factorial(n):
    if n == 0:
        return 1
    else:
        return factorial(n) + sum_of_factorial(n - 1)

$\Theta(n^2)$, we will call factorial $n$ times with arguments $n, n-1, n-2, ..., 0$. The sum from 0 to $n$ is approximately $n^2$.
Video walkthrough

1.2 def fib_recursive(n):
    if n == 0 or n == 1:
        return n
    else:
        return fib_recursive(n - 1) + fib_recursive(n - 2)

$\Theta(\Phi^n)$, where $\Phi$ is the golden ratio. As long as you understand the runtime is exponential in $n$, we would accept your answer.
Video walkthrough

1.3 def fib_iter(n):
    prev, curr, i = 0, 1, 0
    while i < n:
        prev, curr = curr, prev + curr
        i += 1
    return prev

$\Theta(n)$, since the while loop executes $n$ times with each iteration taking a constant $\Theta(1)$ time.
Video walkthrough

1.4 def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total

$\Theta(\log(n))$, because our while loop iterates at most $\log(n)$ times, due to $n$ being halved in every iteration.
Video walkthrough
1.5 \texttt{def} \texttt{mod}_7(n):
    \textbf{if} \ n \ \%\ 7 == 0:
    \texttt{return} \ 0
    \textbf{else}:
    \texttt{return} 1 + \texttt{mod}_7(n - 1)

\(\Theta(1)\), since at worst it will require 6 recursive calls to reach the base case. So this is \(\Theta(6)\), which can be reduced to \(\Theta(1)\).

1.6 \texttt{def} \texttt{bar}(n):
    \textbf{if} \ n \ \%\ 2 == 1:
        \texttt{return} n + 1
    \texttt{return} n

\texttt{def} \texttt{foo}(n):
    \textbf{if} \ n < 1:
        \texttt{return} 2
    \textbf{if} \ n \ \%\ 2 == 0:
        \texttt{return} \texttt{foo}(n - 1) + \texttt{foo}(n - 2)
    \textbf{else}:
        \texttt{return} 1 + \texttt{foo}(n - 2)

What is the order of growth of \texttt{foo(bar(n))}?

\(\Theta(n^2)\)

2 Nonlocal

Until now, you’ve been able to access variables in parent frames, but you have not been able to modify them. The \texttt{nonlocal} keyword can be used to modify a variable in the parent frame outside the current frame. For example, consider \texttt{stepper}, which uses \texttt{nonlocal} to modify \texttt{num}:

\texttt{def} \texttt{stepper}(num):
    \texttt{def} \texttt{step}():
        \texttt{nonlocal} num # declares num as a nonlocal variable
        num = num + 1 # modifies num in the stepper frame
        \texttt{return} num
    \texttt{return} \texttt{step}

However, there are two important caveats with \texttt{nonlocal} variables:

- \textbf{Global variables} cannot be modified using the \texttt{nonlocal} keyword.
- \textbf{Variables in the current frame} cannot be overridden using the \texttt{nonlocal}
keyword. This means we cannot have both a local and nonlocal variable with the same names in a single frame.
Questions

2.1 Draw the environment diagram for the following code.

```python
def stepper(num):
    def step():
        nonlocal num
        num = num + 1
        return num
    return step

s = stepper(3)
s()
s()
```

Video walkthrough
2.2 (Fall 2016) Draw the environment diagram for the following code.

```python
lamb = 'da'
def da(da):
    def lamb(lamb):
        nonlocal da
        def da(nk):
            da = nk + ['da']
            da.append(nk[0:2])
            return nk.pop()
        da(lamb)
    return da([[1], 2]) + 3
da(lambda da: da(lamb))
```
2.3 Write a function that updates and prints a value \( x \) based on input functions.

```python
def memory(n):
    
    >>> f = memory(10)
    >>> f = f(lambda x: x * 2)
    20
    >>> f = f(lambda x: x - 7)
    13
    >>> f = f(lambda x: x > 5)
    True
    
    def f(g):
        nonlocal n
        n = g(n)
        print(n)
        return f
    return f
```

Video walkthrough

3 Mutable Lists

Let’s imagine you order a mushroom and cheese pizza from La Val’s, and
that they represent your order as a list.

A couple minutes later, you realize that you really want onions on the pizza.
Based on what we know so far, La Val’s would have to build an entirely new
list to add onions:

```python
>>> pizza2 = pizza1 + ['onions']  # creates a new python list
>>> pizza2
['cheese', 'mushrooms', 'onions']
>>> pizza1  # the original list is unmodified
['cheese', 'mushrooms']
```

But this is silly, considering that all La Val’s had to do was add onions on
top of \( \text{pizza1} \) instead of making an entirely new \( \text{pizza2} \).

Python actually allows you to \textit{mutate} some objects, includings lists and
dictionaries. Mutability means that the object’s contents can be changed.
So instead of building a new \( \text{pizza2} \), we can use \( \text{pizza1}.\text{append}('onions') \)
to mutate \( \text{pizza1} \).
>>> pizza1.append('onions')
>>> pizza1
['cheese', 'mushrooms', 'onions']

Although lists and dictionaries are mutable, many other objects, such as numeric types, tuples, and strings, are immutable, meaning they cannot be changed once they are created. We can use the familiar indexing operator to mutate a single element in a list. For instance lst[4]='hello' would change the fifth element in lst to be the string 'hello'. In addition to the indexing operator, lists have many mutating methods. List methods are functions that are bound to a specific list. Some useful list methods are listed here:

1. append(el) adds el to the end of the list
2. insert(i, el) insert el at index i (does not replace element but adds a new one)
3. remove(el) removes the first occurrence of el in list, otherwise errors
4. pop(i) removes and returns the element at index i
Questions

3.1 Consider the following definitions and assignments and determine what Python would output for each of the calls below if they were evaluated in order. It may be helpful to draw the box and pointers diagrams to the right in order to keep track of the state.

(a) >>> lst1 = [1, 2, 3]
    >>> lst2 = [1, 2, 3]
    >>> lst1 == lst2  # compares each value

    True

(b) >>> lst1 is lst2  # compares references

    False

c) >>> lst2 = lst1
    >>> lst1.append(4)
    >>> lst1

    [1, 2, 3, 4]

d) >>> lst2

    [1, 2, 3, 4]

e) >>> lst1 = lst1 + [5]
    >>> lst1 == lst2

    False

(f) >>> lst1

    [1, 2, 3, 4, 5]

g) >>> lst2

    [1, 2, 3, 4]

(h) >>> lst2 is lst1

    False

Video walkthrough
3.2 Write a function that takes in two values \( x \) and \( el \), and a list, and adds as many \( el \)'s to the end of the list as there are \( x \)'s.

```python
def add_this_many(x, el, lst):
    """ Adds \( el \) to the end of \( lst \) the number of times \( x \) occurs in \( lst \).
    """
    count = 0
    for element in lst:
        if element == x:
            count += 1
    while count > 0:
        lst.append(el)
        count -= 1
```

3.3 Reverse a list \textit{in place}, i.e. mutate the given list itself, instead of returning a new list.

```python
def reverse(lst):
    """ Reverses \( lst \) in place.
    """
    for i in range(len(lst) // 2):
        lst[i], lst[-i - 1] = lst[-i - 1], lst[i]
```

Video walkthrough