1 Orders of Growth

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- **square(1)** requires one primitive operation: \( \times \) (multiplication). **square(100)** also requires one. No matter what input \( n \) we pass into **square**, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 \cdot 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 \cdot 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 \cdot 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>square(( n ))</td>
<td>( n \cdot n )</td>
<td>1</td>
</tr>
</tbody>
</table>

- **factorial(1)** requires one multiplication, but **factorial(100)** requires 100 multiplications. As we increase the input size of \( n \), the runtime (number of operations) increases linearly proportional to the input.

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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 \cdot 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 \cdot 1 \cdot 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 \cdot 99 \cdots 1 \cdot 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>factorial(( n ))</td>
<td>( n \cdot (n-1) \cdots 1 \cdot 1 )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

For expressing complexity, we use what is called big \( \Theta \) (Theta) notation. For example, if we say the running time of a function **foo** is in \( \Theta(n^2) \), we mean that the running time of the process will grow proportionally with the square of the size of the input as it becomes very large.

- **Ignore lower order terms**: If a function requires \( n^3 + 3n^2 + 5n + 10 \) operations with a given input \( n \), then the runtime of this function is in \( \Theta(n^3) \). As \( n \) gets larger, the lower order terms (10, 5\( n \), and 3\( n^2 \)) all become insignificant compared to \( n^3 \).

- **Ignore constants**: If a function requires 5\( n \) operations with a given input \( n \), then the runtime of this function is in \( \Theta(n) \). We are only concerned with how the runtime grows asymptotically with the input, and since 5\( n \) is still asymptotically linear; the constant factor does not make a difference in runtime analysis.
Kinds of Growth

Here are some common orders of growth, ranked from no growth to fastest growth:

- $\Theta(1)$ — constant time takes the same amount of time regardless of input size
- $\Theta(\log n)$ — logarithmic time
- $\Theta(n)$ — linear time
- $\Theta(n \log n)$ — linearithmic time
- $\Theta(n^2), \Theta(n^3), \text{etc.} — \text{polynomial time}$
- $\Theta(2^n), \Theta(3^n), \text{etc.} — \text{exponential time (considered “intractable”; these are really, really horrible)}$

In addition, some programs will never terminate if they get stuck in an infinite loop.

Questions

What is the order of growth for the following functions?

1.1 def sum_of_factorial(n):
    if n == 0:
        return 1
    else:
        return factorial(n) + sum_of_factorial(n - 1)

1.2 def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total

1.3 def mod_7(n):
    if n % 7 == 0:
        return 0
    else:
        return 1 + mod_7(n - 1)
2  Linked Lists

There are many different implementations of sequences in Python. Today, we’ll explore the linked list implementation.

A linked list is either an empty linked list, or a Link object containing a first value and the rest of the linked list.

To check if a linked list is an empty linked list, compare it against the class attribute Link.empty:

```python
if link is Link.empty:
    print('This linked list is empty!')
else:
    print('This linked list is not empty!')
```

Implementation

class Link:
    empty = ()
    def __init__(self, first, rest=empty):
        assert rest is Link.empty or isinstance(rest, Link)
        self.first = first
        self.rest = rest

    def __repr__(self):
        if self.rest:
            rest_str = ', ' + repr(self.rest)
        else:
            rest_str = ''
        return 'Link({0}{1}).format(repr(self.first), rest_str)

@property
    def second(self):
        return self.rest.first

@second.setter
    def second(self, value):
        self.rest.first = value

    def __str__(self):
        string = '<'
        while self.rest is not Link.empty:
            string += str(self.first) + '
            self = self.rest
        return string + str(self.first) + '>'
Questions

2.1 Write a function that takes in a Python list of linked lists and multiplies them element-wise. It should return a new linked list.

If not all of the Link objects are of equal length, return a linked list whose length is that of the shortest linked list given. You may assume the Link objects are shallow linked lists, and that lst_of_lnks contains at least one linked list.

```python
def multiply_lnks(lst_of_lnks):
    ""
    >>> a = Link(2, Link(3, Link(5)))
    >>> b = Link(6, Link(4, Link(2)))
    >>> c = Link(4, Link(1, Link(0, Link(2))))
    >>> p = multiply_lnks([a, b, c])
    >>> p.first
    48
    >>> p.rest.first
    12
    >>> p.rest.rest.rest
    ()
    ""
```

2.2 Write a function that takes a sorted linked list of integers and mutates it so that all duplicates are removed.

```python
def remove_duplicates(lnk):
    ""
    >>> lnk = Link(1, Link(1, Link(1, Link(1, Link(5)))))
    >>> remove_duplicates(lnk)
    >>> lnk
    Link(1, Link(5))
    """
3 Midterm Review

3.1 Write a function that takes a list and returns a new list that keeps only the even-indexed elements of lst and multiplies them by their corresponding index.

```python
def even_weighted(lst):
    """
    >>> x = [1, 2, 3, 4, 5, 6]
    >>> even_weighted(x)
    [0, 6, 20]
    """

    return [______________________________]
```

3.2 The *quicksort* sorting algorithm is an efficient and commonly used algorithm to order the elements of a list. We choose one element of the list to be the **pivot** element and partition the remaining elements into two lists: one of elements less than the pivot and one of elements greater than the pivot. We recursively sort the two lists, which gives us a sorted list of all the elements less than the pivot and all the elements greater than the pivot, which we can then combine with the pivot for a completely sorted list.

First, implement the *quicksort_list* function. Choose the first element of the list as the pivot. You may assume that all elements are distinct.

_Note: in computer science, “sorting” refers to placing elements in order from least to greatest, not putting things in categories_

```python
def quicksort_list(lst):
    """
    >>> quicksort_list([3, 1, 4])
    [1, 3, 4]
    """

    if _________________________________:
        _______________________________

    pivot = lst[0]

    less = _______________________________

    greater = _______________________________

    return _______________________________
```
3.3 Write a function that takes in a list and returns the maximum product that can be formed using nonconsecutive elements of the list. The input list will contain only numbers greater than or equal to 1.

```python
def max_product(lst):
    """Return the maximum product that can be formed using lst without using any consecutive numbers
    >>> max_product([10,3,1,9,2]) # 10 * 9
    90
    >>> max_product([5,10,5,10,5]) # 5 * 5 * 5
    125
    >>> max_product([])
    1
    """
```

3.4 Complete redundant_map, which takes a tree \( t \) and a function \( f \), and applies \( f \) to each node \((2^d)\) times, where \( d \) is the depth of the node. The root has a depth of 0. It should mutate the existing tree rather than creating a new tree.

```python
def redundant_map(t, f):
    """
    >>> double = lambda x: x*2
    >>> tree = Tree(1, [Tree(1), Tree(2, [Tree(1, [Tree(1)])])])
    >>> redundant_map(tree, double)
    >>> print_levels(tree)
    [2] # 1 * 2 ^ (1) ; Apply double one time
    [4, 8] # 1 * 2 ^ (2) ; Apply double two times
    [16] # 1 * 2 ^ (2 * 2) ; Apply double four times
    [256] # 1 * 2 ^ (2 * 3) ; Apply double eight times
    """
    t.label = _________________________________________________
    new_f = ___________________________________________________
```