1 Linked Lists

There are many different implementations of sequences in Python. Today, we'll
explore the linked list implementation.

A linked list is either an empty linked list, or a Link object containing a first value
and the rest of the linked list.

To check if a linked list is an empty linked list, compare it against the class attribute
Link.empty:

```python
if link is Link.empty:
    print('This linked list is empty!')
else:
    print('This linked list is not empty!')
```

### Implementation

```python
class Link:
    empty = ()

def __init__(self, first, rest=empty):
    assert rest is Link.empty or isinstance(rest, Link)
    self.first = first
    self.rest = rest

def __repr__(self):
    if self.rest:
        rest_str = ', ' + repr(self.rest)
    else:
        rest_str = ''
    return 'Link({0}{1}).format(repr(self.first), rest_str)

def __str__(self):
    string = '<'
    while self.rest is not Link.empty:
        string += str(self.first) + ' ' + str(self.first) + '>'
```
Questions

1.1 Write a function that takes in a Python list of linked lists and multiplies them element-wise. It should return a new linked list.

If not all of the Link objects are of equal length, return a linked list whose length is that of the shortest linked list given. You may assume the Link objects are shallow linked lists, and that lst_of_lnks contains at least one linked list.

```python
def multiply_lnks(lst_of_lnks):
    """
    >>> a = Link(2, Link(3, Link(5)))
    >>> b = Link(6, Link(4, Link(2)))
    >>> c = Link(4, Link(1, Link(0, Link(2))))
    >>> p = multiply_lnks([a, b, c])
    >>> p.first
    48
    >>> p.rest.first
    12
    >>> p.rest.rest.rest is Link.empty
    True
    """
```

1.2 Write a function that takes a sorted linked list of integers and mutates it so that all duplicates are removed.

```python
def remove_duplicates(lnk):
    """
    >>> lnk = Link(1, Link(1, Link(1, Link(1, Link(5)))))
    >>> remove_duplicates(lnk)
    >>> lnk
    Link(1, Link(5))
    """
```
2 Orders of Growth

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- **square(1)** requires one primitive operation: * (multiplication). **square(100)** also requires one. No matter what input \( n \) we pass into **square**, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 \cdot 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 \cdot 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 \cdot 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>square(( n ))</td>
<td>( n \cdot n )</td>
<td>1</td>
</tr>
</tbody>
</table>

- **factorial(1)** requires one multiplication, but **factorial(100)** requires 100 multiplications. As we increase the input size of \( n \), the runtime (number of operations) increases linearly proportional to the input.

<table>
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<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 \cdot 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 \cdot 1 \cdot 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 \cdot 99 \cdots 1 \cdot 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>factorial(( n ))</td>
<td>( n \cdot (n-1) \cdots 1 \cdot 1 )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

For expressing complexity, we use what is called big \( \Theta \) (Theta) notation. For example, if we say the running time of a function **foo** is in \( \Theta(n^2) \), we mean that the running time of the process will grow proportionally with the square of the size of the input as it becomes very large.

- **Ignore lower order terms**: If a function requires \( n^3 + 3n^2 + 5n + 10 \) operations with a given input \( n \), then the runtime of this function is in \( \Theta(n^3) \). As \( n \) gets larger, the lower order terms (10, 5n, and 3n^2) all become insignificant compared to \( n^3 \).

- **Ignore constants**: If a function requires 5n operations with a given input \( n \), then the runtime of this function is in \( \Theta(n) \). We are only concerned with how the runtime grows asymptotically with the input, and since 5n is still asymptotically linear; the constant factor does not make a difference in runtime analysis.
Kinds of Growth

Here are some common orders of growth, ranked from no growth to fastest growth:

- $\Theta(1)$ — constant time takes the same amount of time regardless of input size
- $\Theta(\log n)$ — logarithmic time
- $\Theta(n)$ — linear time
- $\Theta(n \log n)$ — linearithmic time
- $\Theta(n^2), \Theta(n^3), \text{ etc.}$ — polynomial time
- $\Theta(2^n), \Theta(3^n), \text{ etc.}$ — exponential time (considered “intractable”; these are really, really horrible)

We can express growth in terms of any function of $n$—for example, $\Theta(n!)$ grows even more quickly than exponential time growth. However, for this class, you should focus on the growth classes above.

In addition, some programs will never terminate if they get stuck in an infinite loop.

Questions

Let’s look at how we can use this language to describe the runtime of some functions involving various recursive data structures.

2.1 def insert_at_end(lnk, x):
   assert(lnk is not Link.empty, "Cannot add to empty linked list")
   while lnk.rest is not Link.empty:
       lnk = lnk.rest
   lnk.rest = Link(x)

What is the growth of this function in terms of $n$, the length of lnk?

2.2 def concatenate(lnk1, lnk2):
   if lnk2 is Link.empty:
       return lnk1
   else:
       insert_at_end(lnk1, lnk2.first)
       return concatenate(lnk1, lnk2.rest)

What is the growth of this function if the length of lnk1 and lnk2 are both $n$?
2.3 def search(t, x):
  if t.label == x:
    return True
  for b in t.branches:
    if search(b, x):
      return True
  return False

What is the growth of this function in terms of \( n \), the number of nodes in the tree?

2.4 def powers_tree(n):
  if n == 0:
    return Tree(0)
  else:
    return Tree(n, [powers_tree(n // 2) for i in range(2)])

What is the height of the tree returned by a call to \texttt{powers_tree(n)} in terms of \( n \)?

Recall that the height of the tree is equal to the length of the largest path from root to leaf.
3 Midterm Review

3.1 Write a function that takes a list and returns a new list that keeps only the even-indexed elements of lst and multiplies them by their corresponding index.

```python
def even_weighted(lst):
    
    >>> x = [1, 2, 3, 4, 5, 6]
    >>> even_weighted(x)
    [0, 6, 20]

    return [______________________________]
```

3.2 The quicksort sorting algorithm is an efficient and commonly used algorithm to order the elements of a list. We choose one element of the list to be the pivot element and partition the remaining elements into two lists: one of elements less than the pivot and one of elements greater than the pivot. We recursively sort the two lists, which gives us a sorted list of all the elements less than the pivot and all the elements greater than the pivot, which we can then combine with the pivot for a completely sorted list.

First, implement the quicksort_list function. Choose the first element of the list as the pivot. You may assume that all elements are distinct.

*Note: in computer science, “sorting” refers to placing elements in order from least to greatest, not putting things in categories*

```python
def quicksort_list(lst):
    
    >>> quicksort_list([3, 1, 4])
    [1, 3, 4]

    if ________________________________:
        ________________________________

    pivot = lst[0]

    less = ________________________________

    greater = ________________________________

    return ________________________________
```
3.3 Write a function that takes in a list and returns the maximum product that can be formed using nonconsecutive elements of the list. The input list will contain only numbers greater than or equal to 1.

```python
def max_product(lst):
    """Return the maximum product that can be formed using lst without using any consecutive numbers
    >>> max_product([10,3,1,9,2]) # 10 * 9
    90
    >>> max_product([5,10,5,10,5]) # 5 * 5 * 5
    125
    >>> max_product([3])
    1
    """
```

3.4 Complete `redundant_map`, which takes a tree `t` and a function `f`, and applies `f` to each node ($2^d$) times, where `d` is the depth of the node. The root has a depth of 0. It should mutate the existing tree rather than creating a new tree.

```python
def redundant_map(t, f):
    """
    >>> double = lambda x: x*2
    >>> tree = Tree(1, [Tree(1), Tree(2, [Tree(1, [Tree(1)])])])
    >>> redundant_map(tree, double)
    >>> print_levels(tree)
    [2] # 1 * 2 ^ (1) ; Apply double one time
    [4, 8] # 1 * 2 ^ (2), 2 * 2 ^ (2) ; Apply double two times
    [16] # 1 * 2 ^ (2 ^ 2) ; Apply double four times
    [256] # 1 * 2 ^ (2 ^ 3) ; Apply double eight times
    """
    t.label = _________________________________________________
    new_f = ___________________________________________________
    _______________________________________________________
```