1 Efficiency

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- `square(1)` requires one primitive operation: * (multiplication). `square(100)` also requires one. No matter what input `n` we pass into `square`, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 · 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 · 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td><code>n</code></td>
<td><code>square(n)</code></td>
<td><code>n · n</code></td>
<td>1</td>
</tr>
</tbody>
</table>

- `factorial(1)` requires one multiplication, but `factorial(100)` requires 100 multiplications. As we increase the input size of `n`, the runtime (number of operations) increases linearly proportional to the input.

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<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 · 1 · 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 · 99 · ... · 1 · 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td><code>n</code></td>
<td><code>factorial(n)</code></td>
<td><code>n · (n-1) · ... · 1 · 1</code></td>
<td><code>n</code></td>
</tr>
</tbody>
</table>

Here are some general guidelines for finding the order of growth for the runtime of a function:

- If the function is recursive or iterative, you can subdivide the problem as seen above:
  - Count the number of recursive calls/iterations that will be made in terms of input size `n`.
  - Find how much work is done per recursive call or iteration in terms of input size `n`.

The answer is usually the product of the above two, but be sure to pay attention to control flow!
• If the function calls helper functions that are not constant-time, you need to take the runtime of the helper functions into consideration.

• We can ignore constant factors. For example $1000000 \times n$ and $n$ steps are both linear.

• We can also ignore smaller factors. For example if $h$ calls $f$ and $g$, and $f$ is Quadratic while $g$ is linear, then $h$ is Quadratic.

• For the purposes of this class, we take a fairly coarse view of efficiency. All the problems we cover in this course can be grouped as one of the following
  
  – Constant: the amount of time does not change based on the input size. Rule: $n \to 2n \Rightarrow t \to t$.
  
  – Logarithmic: the amount of time changes based on the logarithm of the input size. Rule: $n \to 2n \Rightarrow t \to t + k$.
  
  – Linear: the amount of time changes based on the logarithm of the input size. Rule: $n \to 2n \Rightarrow t \to 2t$.
  
  – Quadratic: the amount of time changes based on the logarithm of the input size. Rule: $n \to 2n \Rightarrow t \to 4t$.
  
  – Exponential: the amount of time changes based on the logarithm of the input size. Rule: $n \to n + 1 \Rightarrow t \to 2t$.

Questions

1.1 What is the efficiency of `bonk`?

```python
def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total
```

1.2 Previously, we looked at the `is_prime` function. Here's the code for it:

```python
def is_prime(n):
    if n == 1:
        return False
    k = 2
    while k < n:
        if n % k == 0:
            return False
        k += 1
    return True
```

What is the efficiency of `is_prime`?

*Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.*
1.3 What is the efficiency of mod_7?

```python
def mod_7(n):
    if n % 7 == 0:
        return 0
    else:
        return 1 + mod_7(n - 1)
```