1 Efficiency

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- **square(1)** requires one primitive operation: $\ast$ (multiplication). **square(100)** also requires one. No matter what input $n$ we pass into **square**, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 $\cdot$ 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 $\cdot$ 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 $\cdot$ 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>square($n$)</td>
<td>$n \cdot n$</td>
<td>1</td>
</tr>
</tbody>
</table>

- **factorial(1)** requires one multiplication, but **factorial(100)** requires 100 multiplications. As we increase the input size of $n$, the runtime (number of operations) increases linearly proportional to the input.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 $\cdot$ 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 $\cdot$ 1 $\cdot$ 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 $\cdot$ 99 $\cdot$ 1 $\cdot$ 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>factorial($n$)</td>
<td>$n \cdot (n-1) \cdot \cdots \cdot 1 \cdot 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Here are some general guidelines for finding the order of growth for the runtime of a function:

- If the function is recursive or iterative, you can subdivide the problem as seen above:
  
  - Count the number of recursive calls/iterations that will be made in terms of input size $n$.
  
  - Find how much work is done per recursive call or iteration in terms of input size $n$.

The answer is usually the product of the above two, but be sure to pay attention to control flow!
Efficiency

• If the function calls helper functions that are not constant-time, you need to take the runtime of the helper functions into consideration.

• We can ignore constant factors. For example 1000000$n$ and $n$ steps are both linear.

• We can also ignore smaller factors. For example if $h$ calls $f$ and $g$, and $f$ is Quadratic while $g$ is linear, then $h$ is Quadratic.

• For the purposes of this class, we take a fairly coarse view of efficiency. All the problems we cover in this course can be grouped as one of the following
  – Constant: the amount of time does not change based on the input size. Rule: $n \rightarrow 2n$ means $t \rightarrow t$.
  – Logarithmic: the amount of time changes based on the logarithm of the input size. Rule: $n \rightarrow 2n$ means $t \rightarrow t + k$.
  – Linear: the amount of time changes based on the logarithm of the input size. Rule: $n \rightarrow 2n$ means $t \rightarrow 2t$.
  – Quadratic: the amount of time changes based on the logarithm of the input size. Rule: $n \rightarrow 2n$ means $t \rightarrow 4t$.
  – Exponential: the amount of time changes based on the logarithm of the input size. Rule: $n \rightarrow n + 1$ means $t \rightarrow 2t$.

Questions

1.1 What is the efficiency of bonk?

```python
def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total
```

Logarithmic, because our while loop iterates at most $\log(n)$ times, due to $n$ being halved in every iteration. Another way of looking at this if you duplicate the input, we only add a single iteration to the time, which also indicates logarithmic. Video walkthrough

1.2 Previously, we looked at the is_prime function. Here’s the code for it:

```python
def is_prime(n):
    if n == 1:
        return False
    k = 2
    while k < n:
        if n % k == 0:
            return False
        k += 1
    return True
```

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.
What is the efficiency of \texttt{is\_prime}?

Linear

1.3 What is the efficiency of \texttt{mod\_7}?

\begin{verbatim}
def mod_7(n):
    if n % 7 == 0:
        return 0
    else:
        return 1 + mod_7(n - 1)
\end{verbatim}

Constant, since at worst it will require 6 recursive calls to reach the base case.