1 Efficiency

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

• square(1) requires one primitive operation: * (multiplication). square(100) also requires one. No matter what input $n$ we pass into square, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2 · 2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100 · 100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>square($n$)</td>
<td>$n · n$</td>
<td>1</td>
</tr>
</tbody>
</table>

• factorial(1) requires one multiplication, but factorial(100) requires 100 multiplications. As we increase the input size of $n$, the runtime (number of operations) increases linearly proportional to the input.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1 · 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2 · 1 · 1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100 · 99 · · · 1 · 1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>factorial($n$)</td>
<td>$n · (n - 1) · · · 1 · 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Here are some general guidelines for finding the order of growth for the runtime of a function:

• If the function is recursive or iterative, you can subdivide the problem as seen above:
  
  – Count the number of recursive calls/iterations that will be made in terms of input size $n$.
  
  – Find how much work is done per recursive call or iteration in terms of input size $n$.

The answer is usually the product of the above two, but be sure to pay attention to control flow!
• If the function calls helper functions that are not constant-time, you need to take the runtime of the helper functions into consideration.

• We can ignore constant factors. For example 1000000n and n steps are both linear.

• We can also ignore smaller factors. For example if h calls f and g, and f is Quadratic while g is linear, then h is Quadratic.

• For the purposes of this class, we take a fairly coarse view of efficiency. All the problems we cover in this course can be grouped as one of the following

  – Constant: the amount of time does not change based on the input size. Rule: \( n \rightarrow 2n \) means \( t \rightarrow t \).

  – Logarithmic: the amount of time changes based on the logarithm of the input size. Rule: \( n \rightarrow 2n \) means \( t \rightarrow t + k \).

  – Linear: the amount of time changes based on the logarithm of the input size. Rule: \( n \rightarrow 2n \) means \( t \rightarrow 2t \).

  – Quadratic: the amount of time changes based on the logarithm of the input size. Rule: \( n \rightarrow 2n \) means \( t \rightarrow 4t \).

  – Exponential: the amount of time changes based on the logarithm of the input size. Rule: \( n \rightarrow n + 1 \) means \( t \rightarrow 2t \).

Questions

1.1 What is the efficiency of bonk?

```python
def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total
```

Logarithmic, because our while loop iterates at most \( \log(n) \) times, due to \( n \) being halved in every iteration. Another way of looking at this if you duplicate the input, we only add a single iteration to the time, which also indicates logarithmic. Video walkthrough

1.2 Previously, we looked at the is_prime function. Here's the code for it:

```python
def is_prime(n):
    if n == 1:
        return False
    k = 2
    while k < n:
        if n % k == 0:
            return False
        k += 1
    return True
```

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.
What is the efficiency of `is_prime`?

Linear

1.3 What is the efficiency of `mod_7`?

```python
def mod_7(n):
    if n % 7 == 0:
        return 0
    else:
        return 1 + mod_7(n - 1)
```

Constant, since at worst it will require 6 recursive calls to reach the base case.