1 Streams

In Python, we can use iterators to represent infinite sequences (for example, the generator for all natural numbers). However, Scheme does not support iterators. Let’s see what happens when we try to use a Scheme list to represent an infinite sequence of natural numbers:

```scheme
(scm> (define (naturals n)
       (cons n (naturals (+ n 1)))))
naturals
(scm> (naturals 0))
Error: maximum recursion depth exceeded
```

Because `cons` is a regular procedure and both its operands must be evaluated before the pair is constructed, we cannot create an infinite sequence of integers using a Scheme list. Instead, our Scheme interpreter supports streams, which are lazy Scheme lists. The first element is represented explicitly, but the rest of the stream’s elements are computed only when needed. Computing a value only when it’s needed is also known as lazy evaluation.

```scheme
(scm> (define (naturals n)
       (cons-stream n (naturals (+ n 1)))))
naturals
(scm> (define nat (naturals 0))
nat
(scm> (car nat))
0
(scm> (cdr nat))
#[promise (not forced)]
(scm> (car (cdr-stream nat)))
1
(scm> (car (cdr-stream (cdr-stream nat))))
2
```

We use the special form `cons-stream` to create a stream:

```scheme
(cons-stream <operand1> <operand2>)
```

`cons-stream` is a special form because the second operand is not evaluated when evaluating the expression. To evaluate this expression, Scheme does the following:

1. Evaluate the first operand.
2. Construct a promise containing the second operand.
3. Return a pair containing the value of the first operand and the promise.
To actually get the rest of the stream, we must call cdr-stream on it to force the promise to be evaluated. Note that this argument is only evaluated once and is then stored in the promise; subsequent calls to cdr-stream returns the value without recomputing it. This allows us to efficiently work with infinite streams like the naturals example above. We can see this in action by using a non-pure function to compute the rest of the stream:

```scheme
scm> (define (compute-rest n)
   ...> (print 'evaluating!)
   ...> (cons-stream n nil))
compute-rest
scm> (define s (cons-stream 0 (compute-rest 1)))
   s
scm> (car (cdr-stream s))
evaluating!
1
scm> (car (cdr-stream s))
1
```

Here, the expression compute-rest 1 is only evaluated the first time cons-stream is called, so the symbol evaluating! is only printed the first time.

When displaying a stream, the first element of the stream and the promise are displayed separated by a dot (this indicates that they are part of the same pair, with the promise as the cdr). If the value in the promise has not been evaluated by calling cdr-stream, we consider it to be not forced. Otherwise, we consider it forced.

```scheme
scm> (define s (cons-stream 1 nil))
   s
scm> s
(1 . #\[promise (not forced)])
scm> (cdr-stream s) ; nil
() 
scm> s
(1 . #\[promise (forced)])
```

Streams are very similar to Scheme lists in that they are also recursive structures. Just like the cdr of a Scheme list is either another Scheme list or nil, the cdr-stream of a stream is either a stream or nil. The difference is that whereas both arguments to cons are evaluated upon calling cons, the second argument to cons-stream isn’t evaluated until the first time that cdr-stream is called.

Here’s a summary of what we just went over:

- nil is the empty stream
- cons-stream constructs a stream containing the value of the first operand and a promise to evaluate the second operand
- car returns the first element of the stream
- cdr-stream computes and returns the rest of stream

\textit{Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.}
Questions

1.1 What would Scheme display?

As you work through these problems, remember that streams have two important components:

- Lazy evaluation – so the remainder of the stream isn’t computed until explicitly requested.
- Memoization – so anything we compute won’t be recomputed.

The examples here stretch these concepts to the limit. In most practical use cases, you may find you rarely need to redefine functions that compute the remainder of the stream.

\[
\text{scm> } (\text{define } (\text{has-even? } s))
\]

\[
(\text{cond}
\begin{align*}
\text{((null? s) #f)} \\
\text{((even? (car s)) #t)} \\
\text{(else (has-even? (cdr-stream s))})
\end{align*})
\]

\[
\text{has-even?}
\]

\[
\text{scm> } (\text{define } f (\ast \text{ 3 x}))
\]

\[
f
\]

\[
\text{scm> } (\text{define nums } (\text{cons-stream 1 (cons-stream (f 3) (cons-stream (f 5) nil))}))
\]

\[
\text{nums}
\]

\[
\text{scm> nums}
\]

\[
\text{scm> (cdr nums)}
\]

\[
\text{scm> (cdr-stream nums)}
\]

\[
\text{scm> nums}
\]

\[
\text{scm> (define f (x) (* 3 x))}
\]

\[
f
\]

\[
\text{scm> (cdr-stream nums)}
\]

\[
\text{scm> (cdr-stream (cdr-stream nums))}
\]

\[
\text{scm> (has-even? nums)}
\]

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.
Using streams can be tricky! Compare the following two implementations of `filter-stream`, the first is a correct implementation whereas the second is wrong in some way.

What's wrong with the second implementation?

; Correct
(define (filter-stream f s)
  (cond
   ((null? s) nil)
   ((f (car s)) (cons-stream (car s) (filter-stream f (cdr-stream s))))
   (else (filter-stream f (cdr-stream s))))

; Incorrect
(define (filter-stream f s)
  (if (null? s) nil
      (let ((rest (filter-stream f (cdr-stream s))))
        (if (f (car s))
            (cons-stream (car s) rest)
            rest))))

1.3 Write a function `map-stream`, which takes a function `f` and a stream `s`. It returns a new stream which has all the elements from `s`, but with `f` applied to each one.

(define (map-stream f s)

scm> (define evens (map-stream (lambda (x) (* x 2)) nat))
evens
scm> (car (cdr-stream evens))
2

1.4 Write a function `slice` which takes in a stream `s`, a `start`, and an `end`. It should return a Scheme list that contains the elements of `s` between index `start` and `end`, not including `end`. If the stream ends before `end`, you can return `nil`.

(define (slice s start end)

scm> (define nat (naturals 0)); See naturals procedure defined earlier
nat
scm> (slice nat 4 12)
(4 5 6 7 8 9 10 11)
Since streams only evaluate the next element when they are needed, we can combine infinite streams together for interesting results! Use it to define a few of our favorite sequences. We’ve defined the function `combine-with` for you below, as well as an example of how to use it to define the stream of even numbers.

```scheme
(define (combine-with f xs ys)
  (if (or (null? xs) (null? ys))
      nil
      (cons-stream
       (f (car xs) (car ys))
       (combine-with f (cdr-stream xs) (cdr-stream ys))))
)
```

scm> (define evens (combine-with + (naturals 0) (naturals 0)))

```scheme
evens
```

scm> (slice evens 0 10)

```scheme
(0 2 4 6 8 10 12 14 16 18)
```

For these questions, you may use the `naturals` stream in addition to `combine-with`.

i. (define factorials

```scheme
scm> (slice factorials 0 10)

```

```scheme
(1 1 2 6 24 120 720 5040 40320 362880)
```

ii. (define fibs

```scheme
scm> (slice fibs 0 10)

```

```scheme
(0 1 1 2 3 5 8 13 21 34)
```

iii. (Extra for practice) Write `exp`, which returns a stream where the $n$th term represents the degree-$n$ polynomial expantion for $e^x$, which is $\sum_{i=0}^{n} x^i / i!$.

You may use `factorials` in addition to `combine-with` and `naturals` in your solution.

```scheme
(define (exp x)

```

```scheme
scm> (slice (exp 2) 0 5)

```

```scheme
(1 3 5 6.333333333333333 7)
```

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.
We can even represent the sequence of all prime numbers as an infinite stream!

Define a function `sieve`, which takes in a stream of increasing numbers and returns a stream containing only those numbers which are not multiples of an earlier number in the stream. We can define `primes` by sifting all natural numbers starting at 2.

Look online for the **Sieve of Eratosthenes** if you need some inspiration.

**Hint:** You might find using `filter-stream` as defined earlier helpful.

```scheme
(define (sieve s)
  (define primes
    (sieve (naturals 2)))

scm> (slice primes 0 10)
(2 3 5 7 11 13 17 19 23 29)
```

*Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.*
2 Extra Questions

2.1 Write a macro that takes an expression and returns a parameter-less lambda procedure with the expression as its body

(define-macro (make-lambda expr)

(scm> (make-lambda (print 'hi))
(lambda () (print (quote hi))))

(scm> (make-lambda (/ 1 0))
(lambda () (/ 1 0)))

(scm> (define print-3 (make-lambda (print 3)))
print-3)

(scm> (print-3)
3)

2.2 Using the make-lambda macro you defined Question 1, define make-stream, a macro which returns a pair of elements, where the second element is not evaluated until cdr-stream is called on it. Also define the procedure cdr-stream, which takes in a stream returned by make-stream and returns the result of evaluating the second element in the stream pair.

Unlike the streams we've seen in lecture and earlier in discussion, if you repeatedly call cdr-stream on a stream returned by make-stream, you may evaluate an expression multiple times.

(define-macro (make-stream first second)

(define (cdr-stream stream)

(scm> (define a (make-stream (print 1) (make-stream (print 2) nil)))
1
a
(scm> (define b (cdr-stream a))
2
b
(scm> (cdr-stream b)
()