1 Recursion and Tree Recursion

Questions

1.1 What are three things you find in every recursive function?

1) Base Case(s)
2) Way(s) to reduce the problem into a smaller problem of the same type
3) Recursive case(s) that uses the solution of the smaller problem to solve the original (large) problem

1.2 When you write a Recursive function, you seem to call it before it has been fully defined. Why doesn’t this break the Python interpreter?

When you define a function, Python does not evaluate the body of the function.

1.3 Below is a Python function that computes the nth Fibonacci number. Identify the three things it contains as a recursive function (from 1.1).

```python
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```

| \begin{solution}[0.75in] |
| Domain is integers, range is integers. |
| Base Cases: if n == 0: ..., elif n == 1: ... |
| Finding Smaller Problems: finding fib(n - 1), fib(n - 2) |
| Recursive Case: when n is neither 0 nor 1, add together the fib(n - 1) and fib(n - 2) to find fib(n) |

1.4 With the definition of the Fibonacci function above, draw out a diagram of the recursive calls made when fib(4) is called.

```
fib(4)
   /  \ 
  fib(3)  fib(2)
 /  |   |  \ 
 fib(2) fib(1) fib(1) fib(0)
 /  |  
 fib(1) fib(0)
```
1.5 What does the following function `cascade2` do? What is its domain and range?

```python
def cascade2(n):
    print(n)
    if n >= 10:
        cascade2(n//10)
    print(n)
```

Domain is integers, range is None. It takes in a number n and prints out n, then
prints out n excluding the ones digit, then prints n excluding the hundreds digit,
and so on, then back up to the full number.

1.6 Consider an insect in an M by N grid. The insect starts at the bottom left corner,
(0, 0), and wants to end up at the top right corner (M-1, N-1). The insect is only
capable of moving right or up. Write a function `paths` that takes a grid length and
width and returns the number of different paths the insect can take from the start
to the goal. (There is a closed-form solution to this problem, but try to answer it
procedurally using recursion.)

```python
def paths(m, n):
    """
    >>> paths(2, 2)
    2
    >>> paths(117, 1)
    1
    """
    if m == 1 or n == 1:
        return 1
    return paths(m - 1, n) + paths(m, n - 1)
```
1.7 Write a procedure `merge(s1, s2)` which takes two sorted (smallest value first) lists and returns a single list with all of the elements of the two lists, in ascending order. Use recursion.

*Hint:* If you can figure out which list has the smallest element out of both, then we know that the resulting merged list will have that smallest element, followed by the merge of the two lists with the smallest item removed. Don’t forget to handle the case where one list is empty!

```python
def merge(s1, s2):
    """ Merges two sorted lists
    >>> merge([1, 3], [2, 4])
    [1, 2, 3, 4]
    >>> merge([1, 2], [])
    [1, 2]
    """

    if len(s1) == 0:
        return s2
    elif len(s2) == 0:
        return s1
    elif s1[0] < s2[0]:
        return [s1[0]] + merge(s1[1:], s2)
    else:
        return [s2[0]] + merge(s1, s2[1:])
```

1.8 Mario needs to jump over a sequence of Piranha plants, represented as a string of spaces (no plant) and P’s (plant!). He only moves forward, and he can either step (move forward one space) or jump (move forward two spaces) from each position. How many different ways can Mario traverse a level without stepping or jumping into a Piranha plant? Assume that every level begins with a space (where Mario starts) and ends with a space (where Mario must end up):

*Hint:* You can get the ith character in a string `s` by using `s[i]`. For example,

```python
>>> s = 'abcdefg'
>>> s[0]
'a'
>>> s[2]
'c'
```

You can find the total number of characters in a string with the built-in `len` function:

```python
>>> s = 'abcdefg'
>>> len(s)
7
>>> len('')
0
```

```python
def mario_number(level):
```

Return the number of ways that Mario can perform a sequence of steps or jumps to reach the end of the level without ever landing in a Piranha plant. Assume that every level begins and ends with a space.

```python
>>> mario_number(' P P ')  # jump, jump
1
>>> mario_number(' P P ')  # jump, jump, step
1
>>> mario_number(' P P ')  # step, jump, jump
1
>>> mario_number(' P P ')  # step, step, jump, jump or jump, jump, jump
2
>>> mario_number(' P PP ')  # Mario cannot jump two plants
0
>>> mario_number(' ')  # step, jump ; jump, step ; step, step, step
3
>>> mario_number(' P ')  #
9
>>> mario_number(' P P P P P P P P P P P ')  #
180
```

```python
def ways(n):
    if n == len(level)-1:
        return 1
    if n >= len(level) or level[n] == 'P':
        return 0
    return ways(n+1) + ways(n+2)
return ways(0)
```

Don't move on until you have been checked-off.

**Topic Review:**
Make sure you have a good understanding of these topics before getting checked off.

- Identifying the three parts of a recursive function.
• Evaluating the output of a recursive function.

• Using multiple base cases and recursive cases (tree recursion)
2 Higher Order Functions

Questions

2.1 What do lambda expressions do? Can we write all functions as lambda expressions? In what cases are lambda expressions useful?

Lambda expressions create functions. When a lambda expression is evaluated, it produces a function. We often use lambdas to create short anonymous functions that we won’t need for too long.

We can’t write all functions as lambda expressions because lambda functions all have to have ‘return’ statements and they can’t contain very complex multi-line expressions.

2.2 Determine if each of the following will error:

```python
>>> 1/0
|\begin{solution}
Error
\end{solution}|
>>> boom = lambda: 1/0
|\begin{solution}
No error, since we don’t evaluate the body of the lambda when we define it.
\end{solution}|
>>> boom()
|\begin{solution}
Error
\end{solution}|
```

2.3 Express the following lambda expression using a `def` statement, and the `def` statement using a lambda expression.

```python
 pow = lambda x, y: x**y
|\begin{solution}[0.25in]
\begin{verbatim}
def pow(x, y):
    return x**y
\end{verbatim}
\end{solution}|
def foo(x):
    def f(y):
        def g(z):
            return x + y * z
        return g
    return f
```
foo = lambda x: lambda y: lambda z: x + y * z
2.4 Draw Environment Diagrams for the following lines of code

\[
square = \lambda x: x \cdot x
\]
\[
\text{higher} = \lambda f: \lambda y: f(f(y))
\]
\[
\text{higher}(\text{square})(5)
\]

Solution: https://goo.gl/LATqV9

\[
a = (\lambda f, a: f(a))(\lambda b: b \cdot b, 2)
\]

Solution: https://goo.gl/TyriuP
2.5 Write `make_skipper`, which takes in a number `n` and outputs a function. When this function takes in a number `x`, it prints out all the numbers between 0 and `x`, skipping every `n`th number (meaning skip any value that is a multiple of `n`).

```python
def make_skipper(n):
    """
    >>> a = make_skipper(2)
    >>> a(5)
    1
    3
    5
    """

    def skipper(x):
        for i in range(x + 1):
            if i % n != 0:
                print(i)
    return skipper
```

2.6 Write a function that takes in a function `cond` and a number `n` and prints numbers from 1 to `n` where calling `cond` on that number returns `True`.

```python
def keep_ints(cond, n):
    """Print out all integers 1..i..n where cond(i) is true"

    >>> def is_even(x):
    ...     # Even numbers have remainder 0 when divided by 2.
    ...     return x % 2 == 0
    ...     return x % 2 == 0
    ...     return x % 2 == 0
    >>> keep_ints(is_even, 5)
    2
    4
    """

    i = 1
    while i <= n:
        if cond(i):
            print(i)
        i += 1
```

Video walkthrough
2.7 Write a function similar to `keep_ints` like before, but now it takes in a number \( n \) and returns a function that has one parameter \( \text{cond} \). The returned function prints out numbers from 1 to \( n \) where calling \( \text{cond} \) on that number returns \( \text{True} \).

```python
def make_keeper(n):
    """Returns a function which takes one parameter cond and prints out all integers 1..i..n where calling cond(i) returns True."
    
    >>> def is_even(x):
    ...     # Even numbers have remainder 0 when divided by 2.
    ...     return x % 2 == 0
    >>> make_keeper(5)(is_even)
    2
    4
    """

def do_keep(cond):
    i = 1
    while i <= n:
        if cond(i):
            print(i)
        i += 1
    return do_keep
```

Video Walkthrough

Don’t move on until you have been checked-off.

**Topic Review:**

Make sure you have a good understanding of these topics before getting checked off.

- The difference between `lambda` and `def`.
- Higher order lambda functions.
• Analyzing functions that return functions.
Optional

2.8 Write `make_alternator` which takes in two functions, `f` and `g`, and outputs a function. When this function takes in a number `x`, it prints out all the numbers between 1 and `x`, applying the function `f` to every odd-indexed number and `g` to every even-indexed number before printing.

```python
def make_alternator(f, g):
    """
>>> a = make_alternator(lambda x: x * x, lambda x: x + 4)
>>> a(5)
1
6
9
8
25
>>> b = make_alternator(lambda x: x * 2, lambda x: x + 2)
>>> b(4)
2
4
6
6
"""

\begin{solution}\
\begin{verbatim}
def alternator(n):
    i = 1
    while i <= n:
        if i % 2 == 1:
            print(f(i))
        else:
            print(g(i))
        i += 1
return alternator
\end{verbatim}
\end{solution}