INSTRUCTIONS

- You have 2 hours to complete the exam.
- The exam is open book, open notes, closed computer, closed calculator.
- Mark your answers on the exam itself. We will not grade answers written on scratch paper.
# Linked Lists

class Link:
    """A linked list cell."
    >>> L = Link(0, Link(1))
    >>> L.first
    0
    >>> L.rest
    Link(1)
    >>> L.first = 2
    >>> L
    Link(2, Link(1))
    >>> L.rest = Link.empty
    >>> L
    Link(2)
    """
    empty = ()

    def __init__(self, first, rest=empty):
        assert rest is Link.empty or isinstance(rest, Link)
        self.first = first
        self.rest = rest

    def __repr__(self):
        if self.rest is Link.empty:
            return "Link({})".format(self.first)
        else:
            return "Link({}, {})".format(self.first, self.rest)

# Trees

class Tree:
    """A tree node."
    ""

def __init__(self, label, branches=[]):
    for c in branches:
        assert isinstance(c, Tree)
    self.label = label
    self.branches = branches

def is_leaf(self):
    return not self.branches
1. (12 points) Pointers

In the following problems, single boxes are variables that contain pointers, and double boxes are Links (see the definition of Link on page ??). To show that a box contains a pointer to the empty list, draw the box like this:

In parts (a) and (b), add arrows and values to the object skeletons on the right to show the final state of the program. Not all boxes will be used. (For examples of what kinds of box and pointer diagrams we're looking for, you might look at parts (c) and (d) first.)

(a) (3 pt)

```python
listy = Link(0, Link(1))
def nest(L):
    if L is Link.empty:
        return L
    N = nest(L.rest)
    L.first = Link(L.first, N)
    return L.first
linky = nest(listy)
```

(b) (3 pt)

```python
v = Link(0, Link(1, Link(2)))
e = v.rest.rest
e.rest = v.rest
v.rest.rest = v
v.rest = Link.empty
ev = Link.empty
ev = v
linky = nest(listy)
```
(c) (3 pt) Show Python code that will produce the situation shown in the diagram. (An arrow pointing to a Link may be shown as pointing anywhere on the double box for that Link.)

\[
\begin{align*}
v & = \text{Link(None, Link(0, Link(1)))} \\
v \text{.first} & = v \text{.rest} \\
v \text{.rest} \text{.rest} \text{.rest} & = v
\end{align*}
\]

(d) (3 pt) Show Python code that converts the situation shown above the line into that shown below the line. Assume \( n \) is even.

\[
\begin{align*}
v & : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \ldots \rightarrow n-1 \rightarrow n \\
w & : \ldots \\
v & : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \ldots \rightarrow n-1 \rightarrow n
\end{align*}
\]

```python
def split2(L):
    """Assuming that linked list L has even length, breaks L into two-element linked lists, and returns a linked list of those lists."""
    if L is Link.empty:
        return Link.empty
    else:
        result = Link(L, split2(L.rest.rest))
        L.rest.rest = Link.empty
        return result
w = split2(v)
```
2. (6 points) Complexity

(a) (1.5 pt) Indicate which of the following assertions are true by circling the letters for those statements. An assertion such as $\Theta(f(n)) \subseteq \Theta(g(n))$ means “any function that is in $\Theta(f(n))$ is also in $\Theta(g(n))$,” and $\Theta(f(n)) = \Theta(g(n))$ if and only if $\Theta(f(n)) \subseteq \Theta(g(n))$ and $\Theta(g(n)) \subseteq \Theta(f(n))$.

A. If $f(n) \in \Theta(1)$ and $g(n) \in \Theta(1)$, then $\Theta(|f(n)| + |g(n)|) \in \Theta(1)$.

B. If $\Theta(f(n)) = \Theta(g(n))$, and $g(n) > 0$ everywhere, then $f(n)/g(n) \in \Theta(1)$.

C. $\Theta(x^2) \subseteq \Theta(x^3)$.

D. $\Theta(2^x) \subseteq \Theta(2^x + x^2)$.

E. If $f(n) \in \Theta(1000 \cdot x^3)$, then $f(20) > 800$.

(b) (1.5 pt) Consider the function

```python
def num_kinks(L):
    c = 0
    i = 0
    while i < len(L):
        j = i
        while j < len(L):
            while j < len(L):
                if kink(L[i], L[j]):
                    c += 1
                    break
            j += 1
        j += 1
    return c
```

Circle the order of growth that best describes the worst-case execution time (measured by the number of calls to kink) of a call to num_kinks as a function of $N$, the length of L.

A. $\Theta(\log N)$

B. $\Theta(N)$

C. $\Theta(N^2)$

D. $\Theta(N^3)$

E. $\Theta(2^N)$
(c) (1.5 pt) Consider the following function on Trees

```python
def count_subtrees(T, p):
    if p(T.label):
        return 1
    else:
        return sum([count_subtrees(child, p) for child in T.branches])
```

Assuming that the maximum number of children of any node is 3, circle the order of growth that best describes the worst-case execution time (measured by the number of calls to `p`) of a call to `count_subtrees` as a function of `N`, the number of nodes in `T`.

A. $\Theta(\log N)$
B. $\Theta(N)$
C. $\Theta(N^2)$
D. $\Theta(N^3)$
E. $\Theta(3^N)$

(d) (1.5 pt) For the same function as in part (c) above, and again assuming that the maximum number of children of any node is 3, circle the order of growth that best describes the worst-case execution time (measured by the number of calls to `p`) of a call to `count_subtrees` as a function of `H`, the height of `T`.

A. $\Theta(\log H)$
B. $\Theta(H)$
C. $\Theta(H^2)$
D. $\Theta(H^3)$
E. $\Theta(3^H)$

3. (1 points) From the Sum of Human Knowledge

This Renaissance composer, famous for his harmonically innovative madrigals, was also infamous for murdering his wife and her lover and thereafter having himself beaten regularly by one of his servants. Who was he?

Carlo Gesualdo
4. (8 points) OOPs!

For each of the expressions in the table below, write the output displayed by the interactive Python interpreter when the expression is evaluated. The output may have multiple lines. If an error occurs, write “ERROR”. No answer requires more than 3 lines. (It’s possible that all of them require even fewer.)

class Thing:
    id = 0

    def fidget(self, n):
        print(n, "A", self.id)

    def fuss(self, x, n):
        print(n, "B")
        self.fidget(n)
        x.fidget(n)

    def twitch(self, n):
        self.waffle(n)

class Gadget(Thing):
    id = 1

class Whatsit(Gadget):
    def fidget(self, n):
        print(n, "D", self.id)

    def waffle(self, n):
        print(n, "D")

    def fiddle(self, x, n):
        x.waffle(n)

t1 = Thing()
t2 = Gadget()
t3 = Whatsit()
t4 = Whatsit()
t3.id = 2

<table>
<thead>
<tr>
<th>Expression</th>
<th>Interactive Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>t3.fidget(1)</td>
<td>1 D 2</td>
</tr>
<tr>
<td>t4.fidget(2)</td>
<td>2 D 1</td>
</tr>
</tbody>
</table>
| t4.fuss(t1, 3)         | 3 B
|                        | 3 D 1              |
|                        | 3 A 0              |
| t4.fiddle(t4, 4)       | 4 D                |
| t4.fiddle(t1, 5)       | ERROR              |
| Thing.id = 3           |                    |
| t1.fidget(6)           | 6 A 3              |
5. (8 points)  Inflections

Fill in the definition of class `Wrinkles` to conform to its doc comment. You need not use all the lines shown.

class Wrinkles:
    
    """An object that contains a sequence of items and a predicate (true/false
    function) and that, when iterated over, produces adjacent pairs of items
    in the sequence that satisfy the predicate.
    >>> w = Wrinkles([1, 2, 3, 2, 4, 8, 5, 4], lambda x, y: x > y)
    >>> for p in w:
    ...     print(p)
    (3, 2)
    (8, 5)
    (5, 4)
    """

    def __init__(self, L, wrinkle):
        self._L = L
        self._wrinkle = wrinkle

    def __iter__(self):
        for k in range(1, len(self._L)):
            if self._wrinkle(self._L[k - 1], self._L[k]):
                yield (self._L[k - 1], self._L[k])
6. (8 points) Tree Paths

Given a tree, $t$, find the length of the longest downward sequence of node labels in the tree that are increasing consecutive integers. For example, in this tree, the longest such sequence has three labels $(1, 2, 3)$:

As illustrated, the longest sequence can start and end anywhere in the tree, not just the root. (Hint: don’t forget there’s a max function.) [The original skeleton was flawed. [The “solution” below shows what we were expecting, but does not properly take care of all cases where the longest path does not start at the root of the tree. The next page shows a complete solution, based on a different solution.]

```python
def longest_seq(t):
    """The length of the longest downward sequence of nodes in T whose labels are consecutive integers.
    >>> t = Tree(1, [Tree(2), Tree(1, [Tree(2, [Tree(3, [Tree(0)])])])])
    >>> longest_seq(t) # 1 -> 2 -> 3
    3
    >>> t = Tree(1)
    >>> longest_seq(t)
    1
    """
    if t.is_leaf():
        return 1

    max_len = 1
    for b in t.branches:
        if b.label == t.label + 1:
            max_len = max(max_len, 1 + longest_seq(b))
        else:
            max_len = max(max_len, longest_seq(b))
    return max_len
```
Here is a proper solution, based on the corrected skeleton

```python
def longest_seq(tr):
    """The length of the longest downward sequence of nodes in TR whose
    labels are consecutive integers.
    >>> t = Tree(1, [Tree(2), Tree(1, [Tree(2, [Tree(3, [Tree(0)])])])])
    >>> longest_seq(t) # 1 -> 2 -> 3
    3
    >>> t = Tree(1)
    >>> longest_seq(t)
    1
    """
    max_len = 1
    def longest(t):
        """Returns longest downward sequence of nodes starting at T whose
        labels are consecutive integers. Updates max_len to that length,
        if greater."""

        nonlocal max_len

        n = 1
        if not t.is_leaf():

            for b in t.branches:

                L = longest(b)

                if b.label == t.label + 1:

                    n = max(n, L + 1)

                    max_len = max(n, max_len)

        return n
    longest(tr)
    return max_len
```
This page deliberately blank.
7. (8 points) Grafting

We want to insert ("graft") branches from a sequence of trees onto a tree in places where a non-leaf node has fewer than $K$ branches, where $K$ is a parameter. For example, given the list of four trees $G$ created by

$$G = [ \text{Tree}(2, [\text{Tree}(0), \text{Tree}(1)]), \text{Tree}(3), \text{Tree}(4), \text{Tree}(5) ]$$

and the tree $T_1$ shown below, we want $T_2 = \text{graft}(T_1, G, 3)$ to destructively (and without creating any new tree nodes) turn $T_1$ into the tree $T_2$:

```
G:  
    ├── 0  
    │   ├── 1  
    │   │   │  2  
    │   │   │   ├── 3  
    │   │   │   │   ├── 4  
    │   │   │   │   │   └── 5  
    │   │   └── 6  
    │       │  7  
    │       │   ├── 8  
    │       │   │   ├── 9  
    │       │   │   │   └── 10  
    │       └── 11  
    └── 12  

T1:  
    ├── A  
    │   │  B  
    │   │   │  D  
    │   │   │   ├── E  
    │   │   │   │   └── F  
    │   │   │   └── G  
    │   └── H  
    └── J  

T2:  
    ├── A  
    │   │  B  
    │   │   │  D  
    │   │   │   └── E  
    │   │   └── F  
    │       │  2  
    │       │   ├── 3  
    │       │   │   └── 4  
    │       └── 5  
    └── 6  
```

The list of trees ($G$ in the example above) will always have enough items to fill all necessary places. Trees are inserted in postorder (that is, bottom to top, left to right).
Fill in the `graft` function here. You need not use all the lines.

```python
def graft(T, L, k):
    """Returns the Tree created by destructively adding trees from L to non-leaf nodes of Tree T with fewer than K branches. Assume that L has enough items to fill all necessary places. Fill in trees in postorder (bottom to top, left to right).""

    grafts = iter(L)

def do_grafts(tr):

    if not tr.is_leaf():

        for b in tr.branches:

            do_grafts(b)

        while len(tr.branches) < k:

            tr.branches.append(next(grafts))

    do_grafts(T)

return T
```